Exercise 1

1. (a) Let $x^{(1)}, x^{(2)} \in S$. Then

$$A\left((1-\lambda)x^{(1)} + \lambda x^{(2)}\right) = (1-\lambda)A\left(x^{(1)}\right) + \lambda A\left(x^{(2)}\right) = (1-\lambda)b + \lambda b = b.$$

(b) Just take $A = (a_1, \ldots, a_n)$ in (a).

2. Suppose $x, y \in c\ell(C)$. Then there exist sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in C such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. For some $\lambda, 0 \leq \lambda \leq 1$, define $z_n := (1-\lambda)x_n + \lambda y_n$. Then, by convexity of C, $z_n \in C$. Moreover $z_n \to (1-\lambda)x + \lambda y$ as $n \to \infty$. Hence this latter point lies in $c\ell(C)$.

3. Let $\{K_\alpha\}_{\alpha \in A}$ be a family of convex sets, and let $\mathcal{K} := \cap_{\alpha \in A} K_\alpha$. Then, for any $x, y \in \mathcal{K}$ by definition of the intersection of a family of sets, $x, y \in K_\alpha$ for all $\alpha \in A$ and each of these sets is convex. Hence for any $\alpha \in A$, and $\lambda \in [0,1], (1-\lambda)x + \lambda y \in K_\alpha$. Hence $(1-\lambda)x + \lambda y \in \mathcal{K}$.