Solution to Exercise 6

1. (a)

$$\partial f(x) = \begin{cases} \{0\} & \text{if } x \in (-1,1) \\ [-1,0] & \text{if } x = -1 \\ [0,1] & \text{if } x = 1 \\ \{-1\} & \text{if } x \in (-2,-1) \\ \{1\} & \text{if } x \in (1,2) \\ (-\infty,-1] & \text{if } x = -2 \\ [1,+\infty) & \text{if } x = 2 \\ \emptyset & \text{if } x \in (-\infty,-2) \cup (2,+\infty). \end{cases}$$

(b) For $\boldsymbol{x} \neq \boldsymbol{0}$, $\nabla \|\boldsymbol{x}\|_2 = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}$.
At $\boldsymbol{x} = \boldsymbol{0}$, we know that $\boldsymbol{u} \in \partial \|\boldsymbol{x}\|_2$ if

$$\|\boldsymbol{y}\|_2 \geq \|\boldsymbol{0}\|_2 + \langle \boldsymbol{y} - \boldsymbol{0}, \boldsymbol{u} \rangle = \langle \boldsymbol{y}, \boldsymbol{u} \rangle \quad \text{for all } \boldsymbol{y} \in \mathbb{R}^N. \tag{1}$$

We can find $\boldsymbol{u}$ that meet these conditions using the Cauchy-Schwarz inequality. Note that

$$\langle \boldsymbol{y}, \boldsymbol{u} \rangle \leq \|\boldsymbol{y}\|_2 \|\boldsymbol{u}\|_2$$

so (1) will hold when $\|\boldsymbol{u}\|_2 \leq 1$. On the other hand, if $\|\boldsymbol{u}\|_2 > 1$, then for $\boldsymbol{y} = \boldsymbol{u}$, we have

$$\langle \boldsymbol{y}, \boldsymbol{u} \rangle = \|\boldsymbol{y}\|_2^2 > \|\boldsymbol{y}\|_2,$$

and (1) does not hold. Therefore

$$\partial \|\boldsymbol{x}\|_2 = \begin{cases} \{\boldsymbol{u} : \|\boldsymbol{u}\|_2 \leq 1\}, & \boldsymbol{x} = \boldsymbol{0} \\ \boldsymbol{x}, & \boldsymbol{x} \neq \boldsymbol{0} \end{cases}$$

2. By the subgradient inequality, we have

$$f(x) \geq f(\widehat{x}) + s^T(x - \widehat{x}) \quad \text{for all } x \in \operatorname{dom} f$$

Suppose that the subdifferential $\partial f(\widehat{x})$ is unbounded. Let s_k be a sequence of subgradients in $\partial f(\widehat{x})$ with $\|s_k\| \to \infty$.

Since $\hat{x}$ lies in the interior of domain, there exists a $\delta > 0$ such that $\hat{x} + \delta y \in \operatorname{dom} f$ for any $y \in \mathbb{R}^n$. Letting $x = \widehat{x} + \delta \frac{s_k}{\|s_k\|}$ for any k, we have

$$f\left(\widehat{x} + \delta \frac{s_k}{\|s_k\|}\right) \geq f(\widehat{x}) + \delta \|s_k\| \quad \text{for all } k$$

As $k \to \infty$, we have $f\left(\widehat{x} + \delta \frac{s_k}{\|s_k\|}\right) - f(\widehat{x}) \to \infty$. However, this relation contradicts the continuity of f at $\widehat{x}$.

3. (a) The first part of the proof is elementary: $\forall g_1 \in \partial f_1(x), \forall g_2 \in \partial f_2(x)$, we have

$$f_i(y) \geq f_i(x) + \langle g_i, y - x \rangle, i = 1, 2, \forall x, y \in \mathbb{R}^N.$$

Hence,

$$f_{(y)} \geq f(x) + \langle g_2 + g_1, y - x \rangle,$$

Therefore $g_1 + g_2 \in \partial f(x)$.

The second part of the proof is kind of difficult and is optional, please refer to Moreau-Rockafellar Theorem if you are interested. (See, for example, Theorem 2.9 in On subdifferential calculus.)

(b) For any $\boldsymbol{A}^T \boldsymbol{g} \in \boldsymbol{A}^T \partial f(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b})$. Then,

$$f(\boldsymbol{A}\boldsymbol{y} + \boldsymbol{b}) \geq f(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}) + \langle \boldsymbol{A}\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}, \boldsymbol{g} \rangle$$

for all $\boldsymbol{y} \in \mathbb{R}^N$. Hence, $\boldsymbol{A}^T \partial f(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}) \subset \partial h(\boldsymbol{x})$.

On the other hand, for any $\boldsymbol{g}' \in \partial h(\boldsymbol{x})$. Then,

$$h(\boldsymbol{y}) \geq h(\boldsymbol{x}) + \langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{g}' \rangle,$$

for any $\boldsymbol{y} \in \mathbb{R}^N$.

Note that for any $\boldsymbol{A}^T \boldsymbol{g} \in \boldsymbol{A}^T \partial f(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b})$,

$$h(\boldsymbol{y}) \geq h(\boldsymbol{x}) + \left\langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{A}^T \boldsymbol{g} \right\rangle.$$

Then $g' \in \boldsymbol{A}^T \partial f(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b})$ and $\partial h(\boldsymbol{x}) = \boldsymbol{A}^T \partial f(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b})$.