# Exercise 6

1. Give the subdifferential sets of the following functions:

(a) $f : \mathbb{R} \to (-\infty, +\infty]$, defined by

$$f(x) := \begin{cases} 0 & \text{if } x \in [-1, +1] \\ |x| - 1 & \text{if } x \in [-2, -1) \cup (1, 2] \\ +\infty & \text{if } x \in (-\infty, -2) \cup (2, +\infty) \end{cases}$$

($\partial f(x)$ is $\emptyset$ for all $x \notin \mathrm{dom}(f)$. Hint: you can draw the image of $f$ and consider the geometric intuition of subgradient.)

(b) $f(x) = \|\boldsymbol{x}\|_2$.

2. Prove that if $\widehat{x} \in \mathrm{int}(\mathrm{dom}\, f)$, then $\partial f(\widehat{x})$ is bounded. (Hint: recall that a convex function is continuous over the interior of its domain. And you can google for more information about subdifferential.)

3. Assume that all functions below are well-defined on all of $\mathbb{R}^N$. Prove the following properties of subdifferential:

(a)If $f(\boldsymbol{x}) = f_1(\boldsymbol{x}) + f_2(\boldsymbol{x})$, then

$$\partial f(\boldsymbol{x}) = \partial f_1(\boldsymbol{x}) + \partial f_2(\boldsymbol{x}).$$

(b) If $h(\boldsymbol{x}) = f(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b})$, then

$$\partial h(\boldsymbol{x}) = \boldsymbol{A}^{\mathrm{T}} \partial f(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}).$$