Exercise 5

1. (a) Prove that a function $f : \mathcal{X} \to [-\infty, +\infty]$ is convex if and only if its epigraph $\mathrm{epi}(f)$ is a convex set.

(b) Prove that if $(f_i)_{i \in I}$ is a family of convex functions $\mathcal{X} \to [-\infty, +\infty]$, with I any set of indices, then $\sup_{i \in I} f_i$ is convex. (Hint: using (a) and the fact that the intersection of any convex sets is still convex, you can give a very simple proof.)

2. Prove that the following operations preserve convexity:

(a) Composition with an affine function: If $f : \mathbb{R}^N \to \mathbb{R}$ is convex, then $g : \mathbb{R}^D \to \mathbb{R}$ defined by

$$g(\boldsymbol{x}) = f(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}),$$

where $\boldsymbol{A} \in \mathbb{R}^{N \times D}$ and $b \in \mathbb{R}^N$, is convex.

(b) Composition with scalar functions: Consider the function $f(\boldsymbol{x}) = h(g(\boldsymbol{x}))$, where $g : \mathbb{R}^N \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$. f is convex if g is convex and h is convex and non-decreasing.

3. Assume $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is twice differentiable. Determine which of the following implications are true. You do NOT need to explain why the implication holds, but please give a counter example if one direction does not hold.

(a) $\nabla^2 f(x) \succ 0, \forall x \in \Omega \Leftrightarrow f$ is strictly convex on Ω.

(b) A function f is strictly convex on $\Omega \subseteq \mathbb{R}^n$ if and only if

$$f(y) > f(x) + \nabla f^T(x)(y - x), \forall x, y \in \Omega, x \neq y$$

(c) f is strongly convex if and only if there exists $m > 0$ such that

$$f(y) \geq f(x) + \nabla^T f(x)(y - x) + m\|y - x\|^2, \forall x, y \in \mathrm{dom}(f),$$

(d) f is strongly convex if and only if there exists $m > 0$ such that

$$\nabla^2 f(x) \succeq mI, \forall x \in \mathrm{dom}(f).$$