Exercise 4

1. (a) Prove that $|x|^\alpha$ is convex on $\mathbb{R}$ for $\alpha \geq 1$.
   (b) Prove that if $f_1, f_2 : \mathbb{R}^N \to \mathbb{R}$ are convex, then $\max(f_1, f_2)$ is convex.

2. Prove the following implication: strong convexity $\Rightarrow$ strict convexity $\Rightarrow$ convexity. Is the converse of either implication is true? Why?

3. (a) Prove that the Quadratic-Over-Linear function $f(x_1, x_2) = \frac{x_1^2}{x_2}$ is convex on $\mathbb{R} \times (0, \infty)$.
   (b) Prove that the quadratic function

$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x} + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{x} + r,$$

where $\boldsymbol{P}$ is symmetric, is convex if and only if $\boldsymbol{P}$ is positive semi-definite.