Exercise 3

1. (a) Give the interior and relative interior of the following sets:
(i) $I^2 := \{(x, y, 0) | 0 \leq x, y \leq 1\}$; (ii) $I^3 := \{(x, y, z) | 0 \leq x, y, z \leq 1\}$.
(b) It is true that if $T \subset S$, then $\text{int}(T) \subset \text{int}(S)$. Does this result hold for relative interior? Explain why.

2. Prove the Prolong Lemma:
Let C be a nonempty convex set. Then we have

$$x \in \text{ri}(C) \Longleftrightarrow \forall \bar{x} \in C, \exists \gamma > 0 \text{ such that } x + \gamma(x - \bar{x}) \in C.$$

In other words, x is a relative interior point iff every line segment in C having x as one of the endpoints can be prolonged beyond x without leaving C. (Hint: Use Line Segment Property of relative interior. You can also Google for more details and applications about this lemma.)

3. Let $f_1, f_2, \ldots, f_k$ be convex functions and $w_1, w_2, \ldots, w_k$ be nonnegative real numbers. Prove that $f(x) = \sum_{i=1}^{k} w_i f_i(x)$ is a convex function.