TUTORIAL NOTES FOR MATH4220

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1. TYPES OF LINEAR SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS

Recall the general form of the linear second-order partial differential equations in $\mathbb{R}^n$,

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + a_0 u = f, \tag{1.1}$$

the type of (1.1) is determined in the following way: let $A = (a_{ij})$ be a $n \times n$ matrix (without loss of generality, A is assumed to be symmetric), denote P to be the number of positive eigenvalues and Z to be the number of zero eigenvalues, then (1.1) is said to be

- hyperbolic if $Z = 0$ and $P = 1$ or $Z = 0$ and $P = n - 1$;
- parabolic if $Z > 0$;
- elliptic if $Z = 0$ and $P = n$ or $Z = 0$ and $P = 0$;
- ultra-hyperbolic if $Z = 0$ and $1 < P < n - 1$.

In particular, for the case $n = 2$,

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1 u_x + a_2 u_y + a_0 u = f, \tag{1.2}$$

we can have a much clear description, i.e. there exists a locally bijective coordinate transformation, $\xi = \phi(x, y)$, $\eta = \psi(x, y)$ which transforms (1.2) into one of the following canonical forms:

(i) If $a_{12}^2 - a_{11}a_{22} > 0$ (hyperbolic),

$$u_{\xi\xi} - u_{\eta\eta} + A_1 u_\xi + A_2 u_\eta + A_0 u = F, \tag{1.3}$$

or

$$u_{\xi\eta} + A_1 u_\xi + A_2 u_\eta + A_0 u = F. \tag{1.4}$$

(ii) If $a_{12}^2 - a_{11}a_{12} = 0$ (parabolic),

$$u_{\xi\xi} + A_1 u_\xi + A_2 u_\eta + A_0 u = F. \tag{1.5}$$

(iii) If $a_{12}^2 - a_{11}a_{12} < 0$ (elliptic),

$$u_{\xi\xi} + u_{\eta\eta} + A_1 u_\xi + A_2 u_\eta + A_0 u = F. \tag{1.6}$$

Let us study some examples.

Example 1. Let

$$u_{xx} + 2u_{xy} + 3u_{yy} + 4u_x + 5u_y + 6 = 0. \tag{1.7}$$

Transform it into a canonical form.

Solution. To transform it into a canonical form, let

$$\xi = \sqrt{2}x, \quad \eta = y - x,$$

then

$$\begin{aligned} u_x &= \sqrt{2}u_\xi - u_\eta, \\ u_y &= u_\eta, \\ u_{xx} &= 2u_{\xi\xi} - 2\sqrt{2}u_{\xi\eta} + u_{\eta\eta}, \\ u_{xy} &= \sqrt{2}u_{\xi\eta} - u_{\eta\eta}, \\ u_{yy} &= u_{\eta\eta}, \end{aligned}$$

therefore we have

$$u_{\xi\xi} + u_{\eta\eta} + 2\sqrt{2}u_\xi + \frac{1}{2}u_\eta + 3 = 0.$$

Example 2. Let

$$e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0. \tag{1.8}$$

Transform it into a canonical form.

Solution. To transform it into a canonical form, let

$$\xi = e^{-x} - e^{-y}, \quad \eta = e^{-x} + e^{-y},$$

then

$$\begin{aligned} u_x &= (-u_\xi - u_\eta)e^{-x}, \\ u_y &= (u_\xi - u_\eta)e^{-y}, \\ u_{xx} &= (u_\xi + u_\eta)e^{-x} + (u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})e^{-2x}, \\ u_{xy} &= (-u_{\xi\xi} + u_{\eta\eta})e^{-x-y}, \\ u_{yy} &= (u_\xi - u_\eta)e^{-y} + (u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})e^{-2y}, \end{aligned}$$

therefore we have

$$u_{\eta\eta} + \frac{1}{4}\eta u_\xi + \frac{1}{4}\xi u_\eta = 0.$$

Example 3. (Tricomi equation) Let

$$yu_{xx} + u_{yy} = 0. \tag{1.9}$$

Transform it into canonical form.

Solution. To transform it into a canonical form, for $y > 0$, let

$$\xi = x, \quad \eta = \frac{2}{3}y^{\frac{3}{2}},$$

then

$$\begin{aligned} u_x &= u_\xi, \\ u_y &= u_\eta y^{\frac{1}{2}}, \\ u_{xx} &= u_{\xi\xi}, \\ u_{yy} &= yu_{\eta\eta} + \frac{1}{2}u_\eta y^{-\frac{1}{2}}, \end{aligned}$$

therefore we have

$$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{3\eta} u_\eta = 0.$$

For $y < 0$, let

$$\xi = x - \frac{2}{3}(-y)^{\frac{3}{2}}, \quad \eta = x + \frac{2}{3}(-y)^{\frac{3}{2}},$$

then

$$\begin{aligned} u_x &= u_\xi + u_\eta, \\ u_y &= (u_\xi - u_\eta)(-y)^{\frac{1}{2}}, \\ u_{xx} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \\ u_{yy} &= (u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})(-y) - (u_\xi - u_\eta) \cdot \frac{1}{2}(-y)^{-\frac{1}{2}}, \end{aligned}$$

therefore we have

$$u_{\xi\eta} - \frac{1}{6(\xi - \eta)}(u_\xi - u_\eta) = 0.$$

A Supplementary Problem

Show that if (1.1) has constant coefficients $a_{ij}(i, j = 1, \cdots, n)$, then there exists a linear transformation from $\mathbf{x} = (x_1, \cdots, x_n)$ to $\mathbf{y} = (y_1, \cdots, y_n)$ such that (1.1) can be changed into an equation with respect to the $\mathbf{y}$ variable without any mixed partial derivatives.

For more materials, please refer to [1, 2, 3, 4].

References

[1] S. Alinhac, *Hyperbolic partial differential equations*, Universitext, Springer, Dordrecht, 2009.

[2] L. C. Evans, *Partial differential equations*, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.

[3] Q. Han and F. Lin, *Elliptic partial differential equations*, vol. 1 of Courant Lecture Notes in Mathematics, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997.

[4] W. A. Strauss, *Partial differential equations. An introduction*, John Wiley & Sons, Inc., New York, 1992.

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