TUTORIAL NOTES FOR MATH4220

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1. THE "WELL-POSEDNESS" OF PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations are extremely important for practical uses. Usually, for a physical quantity u in a domain Ω, we have the physical law P which governs the behavior of u, i.e. $P(u) = 0$ together with some conditions on u, and the aim is to solve u. Generally, P is expressed by partial differential operators, therefore we get partial differential equations supplemented with some "constraint" conditions, however, to solve these problems rigorously, we may get the experience that generally there need to be some solvability condition for the problems, for instance, recall the solvability conditions for equations that we have learned so far (e.x. linear systems, ordinary differential equations, etc). Indeed , the choices of the "constraint" conditions can have a great influence on the solution of the problems, mathematically, for a "good" choice, we call the problem is well-posed, i.e. the problems satisfy the following three properties which are proposed by Jacques Hadamard:

(1) Existence: The problem in fact has a solution;
(2) Uniqueness: There is at most one solution;
(3) Stability: Solution depends continuously on the data given in the problem.

Actually, the way to give a proper "constraint" conditions is really a technique thing, and the solution to the problem may behave essentially different according to different "constraint" conditions, let us illustrate this by studying some concrete examples.

Example 1. Let $\Omega^+ = \{(x, y) : x^2 + y^2 < 1, y > 0\}$, suppose $u \in C^2(\Omega^+)$ is a solution to the following problems,

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \quad , (x, y) \in \Omega, \\ u(x, 0) = 0, \quad u_y(x, 0) &= f(x) \quad , -1 < x < 1. \end{aligned} \tag{1.1}$$

then we extend u as

$$\tilde{u}(\xi, \eta) = \begin{cases} u(\xi, \eta) & , (\xi, \eta) \in \Omega^+, \\ -u(\xi, -\eta) & , (\xi, \eta) \in \Omega^-, \end{cases}$$

where $\Omega^- = \{(x, y) : x^2 + y^2 < 1, y < 0\}$, therefore $\tilde{u} \in C^2(D)$ where D is the unit disk. Since D is simply connected, then by the Green's formula, the function

$$\tilde{v}(\xi, \eta) = \int_{(0,0)}^{(\xi, \eta)} -\frac{\partial u}{\partial y}(x, y)dx + \frac{\partial u}{\partial x}(x, y)dy,$$

is well-defined. It can be verified that $(\tilde{u}, \tilde{v})$ satisfies the Riemann-Cauchy equations

$$\frac{\partial \tilde{u}}{\partial \xi} = \frac{\partial \tilde{v}}{\partial \eta}, \qquad \frac{\partial \tilde{u}}{\partial \eta} = -\frac{\partial \tilde{v}}{\partial \xi},$$

therefore $\tilde{u} + i\tilde{v}$ is a analytic functions, which implies that f has to be analytic.

Example 2. Let $a \in \mathbb{R}$ be a constant,

$$\begin{aligned} \frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} &= 0, \quad (x, y) \in \mathbb{R}^2, \\ u(x, ax) &= 0, \quad x \in \mathbb{R}. \end{aligned} \tag{1.2}$$

then it can be verified that $u = F(y - ax)$ is a solution to the problem where $F \in C^1(\mathbb{R})$ is an arbitrary function with $F(0) = 0$, therefore (1.2) can have infinitely many solutions.

Example 3. Let

$$\begin{aligned} \frac{\partial^2 u^n}{\partial x^2} + \frac{\partial^2 u^n}{\partial y^2} = 0 \quad &, (x, y) \in \mathbb{R} \times \mathbb{R}^+, \\ u^n(x, 0) = 0, \quad \frac{\partial u^n(x, 0)}{\partial y} = \frac{\sin(nx)}{n} \quad &, x \in \mathbb{R}, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad &, (x, y) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial y} = 0 \quad &, x \in \mathbb{R}. \end{aligned} \tag{1.4}$$

It can be verified that the two solutions to the (1.3) and (1.4) are $u^n(x, y) = \frac{1}{n^2} \sinh(ny) \sin(nx)$ and $u(x, y) = 0$ respectively, then on the one hand, we have

$$\lim_{n \to \infty} \frac{\sin(nx)}{n} = 0,$$

on the other hand, we have

$$\lim_{n \to \infty} |u^n(x, y) - u(x, y)| = \infty, \forall (x, y) \in \mathbb{R} \times \mathbb{R}^+.$$

More examples can be found in [1, 2, 3, 4].

References

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