TUTORIAL NOTES FOR MATH4220

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1. ENERGY METHOD FOR WAVE EQUATION

Let us use the energy method to study the solution to Cauchy problem (Initial value problem) of wave equation,

$$\partial_t^2 u - \Delta u = 0, \quad \text{in } (t,x) \in (0,T) \times \mathbb{R}^n, \tag{1.1}$$

$$u(0,\cdot) = u_0, \quad \text{on } \mathbb{R}^n, \tag{1.2}$$

$$\partial_t u(0,.) = u_1, \quad \text{on } \mathbb{R}^n, \tag{1.3}$$

where u_0, $u_1 \in C^2(\mathbb{R}^n)$ are known functions with compact supports in $\mathbb{R}^n$.

1.1. Finite-speed propagation property.

Theorem 1. *Let $x_0 \in \mathbb{R}^n$, $t_0 > 0$, denote $K := \{(t,x) : 0 \le t \le t_0, |x-x_0| \le t_0 - t\}$ and $B(t) := \{x : |x - x_0| \le t_0 - t\}$. Given $T > 0$, let $u \in C^2([0,T] \times \mathbb{R}^n)$ be the solution to*

$$\partial_t^2 u - \Delta u = 0.$$

If $u \equiv u_t \equiv 0$ on $\{t = 0\} \times B(0)$, then $u \equiv 0$ within K.

Proof. Let $t \le t_0$, multiplying $\partial_t u$ on both sides of the equation (1.1) and integrating over $B(t)$, we have

$$\int_{B(t)} \partial_t \left(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla u|^2 \right) dx - \int_{B(t)} \nabla \cdot (\partial_t u \nabla u) dS_x = 0.$$

On the one hand, denote $e(t,x) := \frac{1}{2}|\partial_t u(t,x)|^2 + \frac{1}{2}|\nabla u(t,x)|^2$, then

$$\begin{aligned}
&\int_{B(t)} \partial_t e(t,x) dx \\
=&\int_{|y|\le 1} \partial_t e(s, y(t_0 - s) + x_0)(t_0 - s)^n dy \\
=&\int_{|y|\le 1} (\partial_s e(s, y(t_0 - s) + x_0) + y \cdot \nabla_x e(s, y(t_0 - s) + x_0))(t_0 - s)^n dy \\
:=&I_1 + I_2,
\end{aligned}$$

since

$$\begin{aligned}
I_1 =&\frac{d}{ds} \int_{|y|\le 1} e(s, y(t_0 - s) + x_0)(t_0 - s)^n dy + \int_{|y|\le 1} ne(s, y(t_0 - s) + x_0)(t_0 - s)^{n-1} dy \\
=&\frac{d}{dt} \int_{B(t)} e(t,x) dx + \int_{B(t)} \frac{n}{t_0 - t} e(t,x) dx,
\end{aligned}$$

and

$$
\begin{aligned}
I_2 &= \int_{|y|\le 1} y\cdot\nabla_x e(s, y(t_0-s)+x_0)(t_0-s)^n dy \\
&= \int_{B(t)} \frac{x-x_0}{t_0-t}\cdot\nabla e(t,x)dx \\
&= \int_{B(t)} \nabla\cdot\left(e(t,x)\frac{x-x_0}{t_0-t}\right)dx - \int_{B(t)} \frac{n}{t_0-t}e(t,x)dx,
\end{aligned}
$$

therefore

$$
\int_{B(t)} \partial_t e(t,x)dx = \frac{d}{dt}\int_{B(t)} e(t,x)dx + \int_{\partial B(t)} e(t,x)\frac{x-x_0}{t_0-t}\cdot n(t,x)dS_x,
$$

note that the outer nornal vector at $(t,x)\in\partial B(t)$ is $n(t,x)=\frac{x-x_0}{t_0-t}$, then

$$
\begin{aligned}
&\int_{B(t)} \partial_t\left(\frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2\right)dx \\
=&\frac{d}{dt}\int_{B(t)} \frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2 dx + \int_{\partial B(t)} \frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2 dS_x.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_{B(t)} \nabla\cdot(\partial_t u(t,x)\nabla u(t,x))dx \\
=&\int_{\partial B(t)} \partial_t u(t,x)\nabla u(t,x)\cdot n(t,x)dS_x \\
\le&\int_{\partial B(t)} \frac{1}{2}|\partial_t u(t,x)|^2 dS_x + \int_{\partial B(t)} \frac{1}{2}|\nabla u(t,x)\cdot n(t,x)|^2 dS_x.
\end{aligned}
$$

Therefore

$$
\frac{d}{dt}\int_{B(t)} \frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2 dx \le 0,
$$

which implies

$$
\int_{B(t)} \frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2 dx \le \int_{B(0)} \frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2 dx.
$$

The conclusion follows immediately from the above inequality. □

1.2. **Standard energy estimate.**

Theorem 2. *Let $u\in C^2([0,T]\times\mathbb{R}^n)$ be the solution to* (1.1), (1.2) *and* (1.3), *then*

$$
\int_{\mathbb{R}^n} \frac{1}{2}|\partial_t u(t,x)|^2+\frac{1}{2}|\nabla u(t,x)|^2 dx = \int_{\mathbb{R}^n} \frac{1}{2}|u_1(x)|^2+\frac{1}{2}|\nabla u_0(x)|^2 dx.
$$

Proof. Multiplying $\partial_t u$ on both sides of the equation (1.1) and integrate over $\mathbb{R}^n$, we have

$$
\frac{d}{dt}\int_{\mathbb{R}^n} \frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2 dx - \int_{\mathbb{R}^n} \nabla\cdot(\partial_t u\nabla u)dS_x = 0,
$$

then by the finite-speed propagation property, we have

$$
\frac{d}{dt}\int_{\mathbb{R}^n} \frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2 dx = 0,
$$

therefore

$$\int_{\mathbb{R}^n} \frac{1}{2}|\partial_t u(t,x)|^2 + \frac{1}{2}|\nabla u(t,x)|^2 dx = \int_{\mathbb{R}^n} \frac{1}{2}|u_1(x)|^2 + \frac{1}{2}|\nabla u_0(x)|^2 dx.$$

□

1.3. **Improved energy estimate.**

Theorem 3. *Let $u \in C^2([0,T] \times \mathbb{R}^n)$ be the solution to* (1.1)*,* (1.2) *and* (1.3)*, denote $r := |x|$, then for arbitrary $a \in C^1(\mathbb{R})$,*

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{a(r-t)} \left(\frac{1}{2}|\partial_t u(t,x)|^2 + \frac{1}{2}|\nabla u(t,x)|^2\right) dx \\ &+ \int_0^t \int_{\mathbb{R}^n} a'(r-t) \left|\nabla u(t,x) + \frac{x}{r}\partial_t u(t,x)\right|^2 dx dx \\ =& \int_{\mathbb{R}^n} e^{a(r)} \left(\frac{1}{2}|\partial_t u_1(x)|^2 + \frac{1}{2}|\nabla u_0(x)|^2\right) dx. \end{aligned}$$

Proof. Multiplying $e^{a(r-t)}\partial_t u$ on both sides of the equation (1.1) and integrating over $\mathbb{R}^n$, we have

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n} e^{a(r-t)} \left(\frac{1}{2}|\partial_t u(t,x)|^2 + \frac{1}{2}|\nabla u(t,x)|^2\right) dx \\ &- \int_{\mathbb{R}^n} \nabla \cdot \left(e^{a(r-t)}\partial_t u(t,x)\nabla u(t,x)\right) dx \\ &+ \int_{\mathbb{R}^n} e^{a(r-t)}\Bigg[- \left(\frac{1}{2}|\partial_t u(t,x)|^2 + \frac{1}{2}|\nabla u(t,x)|^2\right) \partial_t a(r-t) \\ &+ \partial_t u(t,x)\nabla a(r-t) \cdot \nabla u(t,x)\Bigg] dx = 0, \end{aligned}$$

since

$$\partial_t a(r-t) = -a'(r-t), \quad \nabla a(r-t) = a'(r-t)\frac{x}{r},$$

and

$$\begin{aligned} &a'(r-t)\left(\frac{1}{2}|\partial_t u(t,x)|^2 + \frac{1}{2}|\nabla u(t,x)|^2 + \partial_t u(t,x)\frac{x}{r} \cdot \nabla u(t,x)\right) \\ =& a'(r-t)\left|\nabla u(t,x) + \frac{x}{r}\partial_t u(t,x)\right|^2, \end{aligned}$$

therefore

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n} e^{a(r-t)} \left(\frac{1}{2}|\partial_t u(t,x)|^2 + \frac{1}{2}|\nabla u(t,x)|^2\right) dx \\ &+ \int_{\mathbb{R}^n} e^{a(r-t)}a'(r-t)\left|\nabla u(t,x) + \frac{x}{r}\partial_t u(t,x)\right|^2 dx = 0, \end{aligned}$$

which implies the results. □

1.4. **Morawetz inequality.**

Theorem 4. *Let $u \in C^2([0,T] \times \mathbb{R}^3)$ be the solution to* (1.1)*,* (1.2) *and* (1.3)*, denote $r := |x|$, then*

$$4\pi \int_0^t |u(s,0)|^2 ds + \int_0^t \int_{\mathbb{R}^n} r^{-1}\left(|\nabla u|^2 - \left|\frac{x}{r}\cdot\nabla u\right|^2\right) dx ds \le 2\int_{\mathbb{R}^n} |\partial_t u|^2 + |\nabla u|^2 dx.$$

Proof. Multipliying $\frac{x}{r}\cdot\nabla u$ on both sides of the equation (1.1), then

$$\partial_t\left[\partial_t u\left(\frac{x}{r}\cdot\nabla u\right)\right]+\nabla\cdot\left[\frac{x}{r}\left(\frac{1}{2}|\nabla u|^2-\frac{1}{2}|\partial_t u|^2\right)-\left(\frac{x}{r}\cdot\nabla u\right)\nabla u\right]$$
$$+\frac{1}{r}\left(|\nabla u|^2-\left|\frac{x}{r}\cdot\nabla u\right|^2\right)+\frac{1}{r}(|\partial_t u|^2-|\nabla u|^2)=0.$$

Moreover, multiplying $\frac{1}{r}u$ on both sides of the equation (1.1), then

$$\partial_t\left(\frac{1}{r}u\partial_t u\right)-\nabla\cdot\left(\frac{1}{r}u\nabla u+\frac{1}{2}u^2\frac{x}{r^3}\right)-\frac{1}{r}(|\partial_t u|^2-|\nabla u|^2)=0,$$

Adding the above two identities, we have

$$\partial_t\left[\partial_t u\left(\frac{x}{r}\cdot\nabla u+\frac{1}{r}u\right)\right]+\frac{1}{r}\left(|\nabla u|^2-\left|\frac{x}{r}\cdot\nabla u\right|^2\right)$$
$$+\nabla\cdot\left[\frac{x}{r}\left(\frac{1}{2}|\nabla u|^2-\frac{1}{2}|\partial_t u|^2\right)-\left(\frac{x}{r}\cdot\nabla u\right)\nabla u-\frac{1}{r}u\nabla u-\frac{1}{2}u^2\frac{x}{r^3}\right]=0.$$

Integrating the above differential identity over $\mathbb{R}^3\backslash B_\varepsilon(0)$, where $B_\varepsilon(0)$ is denoted as the sphere in $\mathbb{R}^3$ with radius ε centering at the origin,

$$\frac{d}{dt}\int_{\mathbb{R}^3\backslash B_\varepsilon(0)}\partial_t u(t,x)\left(\frac{x}{r}\cdot\nabla u(t,x)+\frac{1}{r}u(t,x)\right)dx$$
$$+\int_{\mathbb{R}^3\backslash B_\varepsilon(0)}\frac{1}{r}\left(|\nabla u(t,x)|^2-\left|\frac{x}{r}\cdot\nabla u(t,x)\right|^2\right)dx$$
$$+\underbrace{\int_{\partial B_\varepsilon(0)}\left[\frac{x}{\varepsilon}\left(\frac{1}{2}|\nabla u(t,x)|^2-\frac{1}{2}|\partial_t u(t,x)|^2\right)-\left(\frac{x}{\varepsilon}\cdot\nabla u(t,x)\right)\nabla u(t,x)-\frac{1}{\varepsilon}u(t,x)\nabla u(t,x)-\frac{1}{2}|u(t,x)|^2\frac{x}{\varepsilon^3}\right]\cdot(-\frac{x}{\varepsilon})dS_x}_{I_\varepsilon}=0.$$

Since

$$\lim_{\varepsilon\to 0}I_\varepsilon=\lim_{\varepsilon\to 0}\int_{\partial B_1(0)}\left[z\left(\frac{1}{2}|\nabla_x u(t,\varepsilon z)|^2-\frac{1}{2}|\partial_t u(t,\varepsilon z)|^2\right)-\left(z\cdot\nabla_x u(t,\varepsilon z)\right)\nabla_x u(t,\varepsilon z)\right.$$
$$\left.-\frac{1}{\varepsilon}u(t,\varepsilon z)\nabla_x u(t,\varepsilon z)-\frac{1}{2}|u(t,\varepsilon z)|^2\frac{z}{\varepsilon^2}\right]\cdot(-z)\varepsilon^2 dS_z$$
$$=\frac{4\pi}{2}u^2(t,0),$$

therefore by letting ε goes to 0, we have

$$\frac{d}{dt}\int_{\mathbb{R}^3}\partial_t u(t,x)\left(\frac{x}{r}\cdot\nabla u(t,x)+\frac{1}{r}u(t,x)\right)dx$$
$$+\int_{\mathbb{R}^3}\frac{1}{r}\left(|\nabla u(t,x)|^2-\left|\frac{x}{r}\cdot\nabla u(t,x)\right|^2\right)dx+\frac{4\pi}{2}u^2(t,0)=0,$$

integrating the above identities with respect to t over $[0,T]$,

$$\begin{aligned}&\int_{\mathbb{R}^3}\partial_t u(T,x)\left(\frac{x}{r}\cdot\nabla u(T,x)+\frac{1}{r}u(T,x)\right)dx\\&+\int_0^T\int_{\mathbb{R}^3}\frac{1}{r}\left(|\nabla u|^2-\left|\frac{x}{r}\cdot\nabla u\right|^2\right)dxds+\frac{4\pi}{2}\int_0^T u^2(s,0)ds\\=&\int_{\mathbb{R}^3}\partial_t u(0,x)\left(\frac{x}{r}\cdot\nabla u(0,x)+\frac{1}{r}u(0,x)\right)dx.\end{aligned}$$

Since

$$\begin{aligned}&\int_{\mathbb{R}^3}\partial_t u(t,x)\left(\frac{x}{r}\cdot\nabla u(t,x)+\frac{1}{r}u(t,x)\right)dx\\\leq&\frac{1}{2}\int_{\mathbb{R}^3}|\partial_t u(t,x)|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}\left|\frac{x}{r}\cdot\nabla u(t,x)+\frac{1}{r}u(t,x)\right|^2dx,\end{aligned}$$

and

$$\int_{\mathbb{R}^3}2\frac{x}{r}\cdot\nabla u(t,x)\cdot\frac{1}{r}u(t,x)=\int_{\mathbb{R}^3}\frac{x}{r^2}\cdot\nabla u^2(t,x)dx=-\int_{\mathbb{R}^3}\frac{1}{r^2}u^2(t,x)dx,$$

therefore

$$\begin{aligned}&\int_{\mathbb{R}^3}\partial_t u(t,x)\left(\frac{x}{r}\cdot\nabla u(t,x)+\frac{1}{r}u(t,x)\right)dx\\\leq&\frac{1}{2}\int_{\mathbb{R}^3}|\partial_t u(t,x)|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}\left|\frac{x}{r}\cdot\nabla u(t,x)\right|^2dx,\end{aligned}$$

then

$$\begin{aligned}&4\pi\int_0^T u^2(s,0)ds+2\int_0^T\int_{\mathbb{R}^3}\frac{1}{r}\left(|\nabla u|^2-\left|\frac{x}{r}\cdot\nabla u\right|^2\right)dxds\\\leq&\int_{\mathbb{R}^3}|\partial_t u(t,x)|^2+|\partial_t u(0,x)|^2dx+\int_{\mathbb{R}^3}\left|\frac{x}{r}\cdot\nabla u(t,x)\right|^2+\left|\frac{x}{r}\cdot\nabla u(0,x)\right|^2dx,\end{aligned}$$

which implies the results. □

Remark. By moving the origin, we can prove that $\int_0^T|u(s,x)|^2ds$ is uniform bounded. Moreover, by the geometric properties in $\mathbb{R}^3$, arbitrary finite domain can be covered by three sets of nested spheres, also note that $|\nabla u|^2-\left|\frac{x}{r}\cdot\nabla u\right|^2$ represent the tangential derivatives of u along the boundary $\partial B_r(0)$, therefore by using the Morawetz estimate, $\|r^{-\frac{1}{2}}\nabla u\|_{L^2((0,T);L^2(\mathbb{R}^3))}$ can be controlled as well.

A Supplementary Problem

For a bounded region $\Omega\subset\mathbb{R}^n$ and a positive constant T, if u satisfies

$$\begin{aligned}\partial_t^2u-\Delta u=&f,\quad\text{in }\Omega_T,\\u(0,\cdot)=&u_0,\quad\text{on }\Omega,\\\partial_t u(0,.)=&u_1,\quad\text{on }\Omega,\\u=&0,\quad\text{on }[0,T]\times\partial\Omega.\end{aligned}$$

Show that

$$\sup_{0\le t\le T}\int_\Omega |u(t,x)|^2+|\partial_t u(t,x)|^2+|\nabla u(t,x)|^2dx$$
$$\le C\left(\int_\Omega |u_0(x)|^2+|\partial_t u_0(x)|^2+|\nabla u_0(x)|^2dx+\int_0^T\int_\Omega |f(t,x)|^2dxdt\right),$$

where $C=C(T)$ is a positive constant.

For more materials, please refer to [1, 2, 3, 4].

References

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