Mid-term Examination

Partial Differential Equations (MATH4220)
(Academic Year 2022/2023, Second Term)

Date: March 16, 2023. **Time allowed:** 08:30 - 10:15.

1. Consider the following three questions.
   (a) (5 points) State the definition of a well-posed PDE problem.
   (b) (5 points) Is the following problem well-posed? Why?
   $$\begin{cases} \dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} + \dfrac{\mathrm{d}u}{\mathrm{d}x} = 1, \quad x \in (0,1), \\ u'(0) = 1 \quad \text{and} \quad u'(1) = 0. \end{cases}$$
   (c) (10 points) State and prove the uniqueness and continuous dependence of solutions to the problem
   $$\begin{cases} \partial_t u - \partial_x^2 u = 0, \quad (t,x) \in (0,T) \times (0,1), \\ \partial_x u(t,0) = 0, \ \partial_x u(t,1) = 0, \ t \in (0,T), \\ u(t,x)_{|t=0} = \phi(x), \qquad x \in [0,1]. \end{cases}$$
2. Let Ω be a bounded, connected, open set of $\mathbb{R}^3$. We say that $v \in C^2(\bar{\Omega})$ is subharmonic if
   $$-\Delta v \le 0, \quad \text{on } \Omega.$$
   (a) (5 points) Prove for subharmonic function v that
   $$v(x) \le \frac{3}{4\pi r^3} \int_{B_r(x)} v(y) \mathrm{d}y.$$
   (b) (5 points) Prove that therefore $\max_{\bar{\Omega}} v(x) = \max_{\partial\Omega} v(x)$.
3. We shall consider functions $h = h(t,x) : [0,\infty) \times \mathbb{R} \to (0,\infty)$ which are 2π-periodic with respect to x, belong to $C^\infty([0,\infty) \times \mathbb{R})$ and satisfy the 1D heat equation
   $$\partial_t h - \partial_x^2 h = 0, \quad \text{in } (0,\infty) \times \mathbb{R}.$$
   In this problem, we are interested in finding quantities that are non-increasing along the flow of the heat equation. More specifically, we will consider the entropy notions.

(a) (10 points) Show that

$$(\partial_t - \partial_x^2) \log h = \frac{|\partial_x h|^2}{h^2},$$
$$(\partial_t - \partial_x^2) h \log h = -\frac{|\partial_x h|^2}{h},$$
$$(\partial_t - \partial_x^2) \frac{|\partial_x h|^2}{h} = -2h \left| \frac{\partial_x^2 h}{h} - \frac{|\partial_x h|^2}{h^2} \right|^2.$$

(b) (10 points) Introduce the function $t \mapsto H(t)$, known as the Boltzmann's entropy, defined by,

$$H(t) = \int_0^{2\pi} h(t,x) \log(h(t,x)) \mathrm{d}x.$$

Show that the function H decays in a convex manner that is,

$$\frac{\mathrm{d}H}{\mathrm{d}t} \leq 0 \quad \text{and} \quad \frac{\mathrm{d}^2 H}{dt^2} \geq 0.$$

(c) From now on, we assume that the initial data $h_0(x) = h(0,x)$ is a probability density which means that

$$\int_0^{2\pi} h_0(x) \mathrm{d}x = 1.$$

(i) (5 points) Show that $\int_0^{2\pi} h(t,x)\mathrm{d}x = 1$ for all time $t \geq 0$.

(ii) (5 points) Introduce the functions $t \mapsto F(t)$ and $t \mapsto J(t)$ defined by,

$$F(t) = \int_0^{2\pi} \frac{|\partial_x h(t,x)|^2}{h(t,x)} \mathrm{d}x,$$
$$J(t) = \int_0^{2\pi} h(t,x) \left| \frac{\partial_x^2 h(t,x)}{h(t,x)} - \frac{|\partial_x h(t,x)|^2}{h(t,x)^2} \right|^2 \mathrm{d}x.$$

Show that

$$\frac{\mathrm{d}F}{\mathrm{d}t} + 2J = 0, \quad \text{for all } t \geq 0.$$

(iii) (10 points) Given a real number λ introduce the quantity

$$A(\lambda) = \int_0^{2\pi} h(t,x) \left| \frac{\partial_x^2 h(t,x)}{h(t,x)} - \frac{|\partial_x h(t,x)|^2}{h(t,x)^2} + \lambda \right|^2 \mathrm{d}x \geq 0.$$

Show that

$$A(\lambda) = J - 2\lambda F + \lambda^2.$$

Then by choosing λ appropriately deduce that $J \geq F^2$.

(d) (10 points) Consider the function $t \mapsto N(t)$ defined by,

$$N(t) = \exp(-2H(t)), \quad \text{for all } t \geq 0.$$

Show that the function $t \mapsto N(t)$ is concave that is,

$$\frac{\mathrm{d}^2 N}{\mathrm{d}t^2} \leq 0.$$

(e) (10 points) Let $u : [0, \infty) \to (0, \infty)$ be a C^1 function satisfying for some $K > 0$ the inequality

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Ku^2 \leq 0.$$

Show that

$$u(t) \leq \frac{u(0)}{1 + Ku(0)t}, \quad \text{for all } t \geq 0.$$

(f) (10 points) Conclude that there exists a constant $C > 0$ such that, for all time $t \geq 1$,

$$\int_0^{2\pi} \frac{|\partial_x h(t,x)|^2}{h(t,x)} \mathrm{d}x \leq \frac{C}{t}.$$

*************** END OF THE QUESTIONS ***************