Make-up Midterm Examination

Partial Differential Equations (MATH4220)
(Academic Year 2022/2023, Second Term)

Date: March 30, 2023. **Time allowed:** 14:30 - 16:15.

1. Consider the following four questions.

   (a) (5 points) State the definition of a well-posed PDE problem.

   (b) (5 points) What is the type of this equation

   $$\partial_t^2 u + \partial_{tx}^2 u - 2\partial_x^2 u = 0.$$

   (c) (5 points) Solve the problem

   $$\partial_t u + 4\partial_x u - 2u = 0, \quad \text{with } u_{|t=0} = x^2.$$

   (d) (5 points) Find all the solutions to

   $$\partial_x u - 2\partial_y u + 2u = 1.$$

2. Let Ω be a bounded, connected, open set of $\mathbb{R}^3$. We say that $v \in C^2(\bar{\Omega})$ is harmonic if

   $$-\Delta v = 0, \quad \text{on } \Omega.$$

   (a) (10 points) State and prove the mean-value property of harmonic function.

   (b) (10 points) Show that $\max_{\bar{\Omega}} v(x) = \max_{\partial\Omega} v(x)$.

3. We shall consider functions $h = h(t,x) : [0,\infty) \times \mathbb{R} \to (0,\infty)$ which are 2π-periodic with respect to x, belong to $C^\infty([0,\infty) \times \mathbb{R})$ and satisfy the 1D mean-curvature equation

   $$\partial_t h - \frac{\partial_x^2 h}{1 + |\partial_x h|^2} = 0, \quad \text{in } (0,\infty) \times \mathbb{R}.$$

   We are interested in finding quantities that are non-increasing along the solution flow.

(a) (10 points) Show that

$$\partial_x \left(\frac{\partial_x h}{\sqrt{1+|\partial_x h|^2}} \right) = \frac{\partial_x^2 h}{(1+|\partial_x h|^2)^{\frac{3}{2}}},$$

and then deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{2\pi} \sqrt{1+|\partial_x h|^2} \mathrm{d}x \leq 0.$$

Hint: Using the structure of mean-curvature equation, and then integration by parts.

(b) (10 points) Show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{2\pi} |\partial_x h|^2 \mathrm{d}x \leq 0.$$

(c) (i) (10 points) Show that

$$\partial_t h - \frac{\partial_x^2 h}{1+|\partial_x h|^2} = \partial_t h - \partial_x \left(\arctan(\partial_x h) \right),$$

and then deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{2\pi} h^2 \mathrm{d}x \leq 0.$$

(ii) (10 points) Show that

$$\partial_t \left((\partial_x h) \arctan(\partial_x h) \right) = \left(\arctan(\partial_x h) + \frac{\partial_x h}{1+|\partial_x h|^2} \right) \partial_x \left(\frac{\partial_x^2 h}{1+|\partial_x h|^2} \right),$$

and then deduce that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_0^{2\pi} h^2 \mathrm{d}x \geq 0.$$

(iii) (10 points) Show that

$$\partial_t^2 h = \partial_x \left(\frac{\partial_{tx} h}{1+|\partial_x h|^2} \right).$$

(iv) (10 points) Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{2\pi} |\partial_t h|^2 \mathrm{d}x \leq 0,$$

and then deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{2\pi} \left(\frac{\partial_x^2 h}{1+|\partial_x h|^2} \right)^2 \mathrm{d}x \leq 0.$$

*************** END OF THE QUESTIONS ***************