Homework IV
Due Date: 10/05/2023

Exercise 1 (3 points). Compute the following three Fourier series.
(i) Find the Fourier sine series of $\phi(x) = x$ on the interval $[0, \pi]$.
(ii) Find the Fourier cosine series of $\phi(x) = x$ on the interval $[0, \pi]$.
(iii) Find the full Fourier series of $\phi(x) = x$ on the interval $[-\pi, \pi]$.

Exercise 2 (2 points). Solve the following problem on $[0, \infty) \times [0, \pi] \subset \mathbb{R}_t \times \mathbb{R}_x$,

$$\begin{cases} \partial_t^2 u = \partial_x^2 u, \\ u(t,0) = u(t,\pi) = 0, \\ u(0,x) = x, \ \partial_t u(0,x) = 0. \end{cases}$$

Exercise 3 (3 points). Consider the following two questions.
(i) Let f be the function defined on $[-\pi, \pi]$ by $f(x) = |x|$. Using Parseval's identity to show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(ii) Consider the 2π-periodic odd function defined on $[0, \pi]$ by $f(x) = x(\pi - x)$. Show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

Exercise 4 (2 points). In this question, our goal is to show that the solution flow of the incompressible Euler system on the torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ is not uniformly continuous in $H^1(\mathbb{T}^2)$. We start with some notations and conventions.

Notations and conventions. The 2-torus $\mathbb{T}^2$ is the cube $[0, 2\pi]^2$ with opposite sides identified. This means that the points $(x_1, 0)$ and $(x_1, 2\pi)$ are identified and the points $(0, x_2)$ and $(2\pi, x_2)$ are identified. A function $f : \mathbb{T}^2 \to \mathbb{C}$ is a function $f : \mathbb{R}^2 \to \mathbb{C}$ which is 2π-periodic in (x_1, x_2), that is, $f(x_1, x_2) = f(x_1 + 2\pi, x_2) = f(x_1, x_2 + 2\pi)$. We define $L^1(\mathbb{T}^2)$ as the space of f such that

$$\|f\|_{L^1(\mathbb{T}^2)} := \int_0^{2\pi} \int_0^{2\pi} |f(x_1, x_2)| \mathrm{d}x_1 \mathrm{d}x_2 < \infty.$$

From now on, we denote $x = (x_1, x_2) \in [0, 2\pi]^2$ and $k = (k_1, k_2) \in \mathbb{Z}^2$. To simplify notation, we also denote

$$\int_{[0,2\pi]^2} f(x)\mathrm{d}x := \int_0^{2\pi} \int_0^{2\pi} |f(x_1, x_2)| \mathrm{d}x_1 \mathrm{d}x_2, \quad \text{for any } f \in L^1(\mathbb{T}^2).$$

To $f \in L^1(\mathbb{T}^2)$, we associate its Fourier coefficient,

$$(\mathcal{F}f)(k) := \widehat{f}(k) = \int_{[0,2\pi]^2} e^{-ik\cdot x} f(x)\mathrm{d}x, \quad \text{for } k \in \mathbb{Z}^2.$$

Then we define $H^1(\mathbb{T}^2)$ as the space of f such that

$$\|f\|_{H^1(\mathbb{T}^2)}^2 := \sum_{k\in\mathbb{Z}^2} (1 + |k|^2)|\widehat{f}(k)|^2 < \infty.$$

Eventually, if $U = (u_1, u_2)$, we set $\|U\|_{H^1(\mathbb{T}^2)}^2 = \|u_1\|_{H^1(\mathbb{T}^2)}^2 + \|u_2\|_{H^1(\mathbb{T}^2)}^2$.

The Euler system on $U = (u_1, u_2)$ is

$$\begin{aligned} &\partial_t U + (U \cdot \nabla_x)U - \nabla_x \Delta^{-1} \mathrm{div}((U \cdot \nabla_x)U) = 0, \quad \text{for } (t,x) \in [0,\infty) \times \mathbb{T}^2, \\ &\mathrm{div} U = 0, \quad \text{for } (t,x) \in [0,\infty) \times \mathbb{T}^2 \quad \text{and} \quad U_{|t=0} = U^0, \quad \text{for } x \in \mathbb{T}^2. \end{aligned} \tag{1}$$

Here, the operator Δ^{-1} is the inverse operator of Δ, that is,

$$\widehat{\Delta^{-1}v}(k) = -\frac{1}{|k|^2}\widehat{v}(k), \quad \text{for any } k \in \mathbb{Z}^2.$$

The goal of the following questions is to construct two sequences of initial data $\left(U_n^0, V_n^0\right)_{n\in\mathbb{N}^+}$ of the problem (1) such that

$$\lim_{n\to\infty} \left\|U_n^0 - V_n^0\right\|_{H^1(\mathbb{T}^2)} = 0, \tag{2}$$

but, for $t > 0$ arbitrary small and for n large enough,

$$\|U_n(t) - V_n(t)\|_{H^1(\mathbb{T}^2)} \geq c_0(t) > 0, \tag{3}$$

where U_n and V_n denote the solutions of the Euler system (1) with initial data U_n^0 and V_n^0 at $t = 0$, respectively.

1. Compute the norms in $H^1(\mathbb{T}^2)$ of the functions $f_1(x_1, x_2) = C \in \mathbb{C}$, $f_2(x_1, x_2) = \sin(nx_1)$, and $f_3(x_1, x_2) = \sin(nx_2)$ where $n \in \mathbb{Z}$.

For $n \in \mathbb{N}$ and $\omega \in \mathbb{R}$, we set $U_{n,\omega} = (u_1, u_2)$ where

$$\begin{aligned} u_1(t,x) &= (\omega + \cos(nx_2 - \omega t))\, n^{-1}, \\ u_2(t,x) &= (\omega + \cos(nx_1 - \omega t))\, n^{-1}. \end{aligned} \tag{4}$$

2. (a) Compute

$$\mathrm{div} U_{n,\omega}, \quad \partial_t U_{n,\omega} + (U_{n,\omega}\cdot\nabla_x)U_{n,\omega}, \quad \text{and} \quad \mathrm{div}\left((U_{n,\omega}\cdot\nabla_x)U_{n,\omega}\right).$$

(b) Set $\omega_n(t,x) = \sin(nx_1 - \omega t)\sin(nx_2 - \omega t)$. Show that

$$\Delta\left(\frac{1}{2n^2}\omega_n\right) = -\omega_n, \quad \text{for any } n \in \mathbb{N}.$$

(c) Let Q be a periodic solution of equation $\Delta Q = \omega_n$ and let $R = Q + \frac{1}{2n^2}\omega_n$. Show that $\Delta R = 0$.

(d) Using the inverse Fourier transform, we have

$$R(x) = \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}^2} \widehat{R}(k)e^{ik\cdot x}, \quad \text{for } x \in [0, 2\pi]^2.$$

Show that

$$R(x) = \frac{1}{(2\pi)^2}\widehat{R}(0) \quad \text{and so} \quad Q = \Delta^{-1}\omega_n = -\frac{1}{2n^2}\omega_n + C_n, \quad \text{where } C_n \in \mathbb{C}.$$

Hint: Using the fact that

$$\widehat{\Delta R}(k) = -|k|^2\widehat{R}(k), \quad \text{for any } k \in \mathbb{Z}^2.$$

(e) Compute $\nabla_x\Delta^{-1}\mathrm{div}\left((U_{n,\omega}\cdot\nabla_x)U_{n,\omega}\right)$.

(d) Deduce that $U_{n,\omega}$ is a solution of (1).

Let

$$U_{n,\omega}^0 = U_{n,\omega}(0) = \left((\omega + \cos(nx_2))\, n^{-1}, (\omega + \cos(nx_1))\, n^{-1}\right).$$

Denote $U_n^0 = U_{n,1}^0$ and $V_n^0 = V_{n,-1}^0$.

3. Show that (2) is satisfied.

4. Let U_n and V_n be the solutions of Euler system (1) corresponding to these initial data. Show that, there exist $C_3 > 0$ and $C_4 > 0$ such that

$$\|U_n - V_n\|_{H^1(\mathbb{T}^2)} \geq C_3|\sin t| - \frac{C_4}{n},$$

and then conclude that (3) is true.

Hint: From the definition of U_n and V_n to compute $U_n - V_n$, and then using the fact that $\cos a - \cos b = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$ to rewrite $U_n - V_n$.