HOMEWORK II (DUE DATE: 10/04/2023)

Exercise 1 (2 points)**.** Find the Green's function for $\Delta u = 0$ on the following domain Ω.

(1) $\Omega = \{(x,y) \in \mathbb{R}^2 : -\infty < x < \infty, 0 < y < \infty\}$.
(2) $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < \infty, 0 < y < \infty\}$.

Exercise 2 (2 points)**.** Find the Green's function for $u_t - \Delta u = 0$ on the following domain Ω.

(1) $\Omega = \{x \in \mathbb{R} : 0 < x < l\}$.
(2) $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < \infty, 0 < y < \infty\}$.

Exercise 3 (2 points)**.** Solve the following problems:

$$(1) \begin{cases} u_t(t,x) = u_{xx}(t,x) + u(t,x), & (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = \phi(x), & x \in \mathbb{R}. \end{cases}$$

$$(2) \begin{cases} u_t(t,x) = u_{xx}(t,x) + u_x(t,x), & (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = \phi(x), & x \in \mathbb{R}. \end{cases}$$

Exercise 4 (2 points)**.** Let $u \in C^{1,2}((0,\infty) \times (0,1)) \cap C([0,\infty) \times [0,1])$ be the solution to the following problems,

$$\begin{aligned} u_t(t,x) - u_{xx}(t,x) = 0, &\quad (t,x) \in (0,\infty) \times (0,1), \\ u(0,x) = x(1-x), &\quad x \in [0,1], \\ u(t,0) = u(t,1) = 0, &\quad t \in [0,\infty). \end{aligned}$$

Prove

(1) u is non-negative.
(2) u decays to 0 uniformly as time goes to infinity. (Hints: show that $u(t,x) \leq x(1-x)e^{-t}$.)

Exercise 5 (2 points)**.** Let $u \in C^{1,2}((0,T) \times \mathbb{R}) \cap C([0,T] \times \mathbb{R})$ be the solution to the following problems,

$$u_t(t,x) - u_{xx}(t,x) = 0, \quad (t,x) \in (0,T) \times \mathbb{R}, \tag{5.1}$$

$$u(0,x) = \phi(x), \quad x \in \mathbb{R}, \tag{5.2}$$

and there exists two constants c, $C > 0$ such that

$$|u(t,x)| \leq Ce^{cx^2}, \quad (t,x) \in (0,T) \times \mathbb{R}. \tag{5.3}$$

Prove

(1) u is the unique solution to the Cauchy problem (5.1), (5.2) satifying (5.3) and

$$\sup_{(0,T)\times\mathbb{R}} |u| \leq \sup_{\mathbb{R}} |\phi|.$$

(2) Denote
$$\tilde{u}(t,x) = \sum_{k=0}^{\infty} \frac{d^k \varphi(t)}{dt^k} \frac{x^{2k}}{(2k)!},$$
where
$$\varphi(t) = \begin{cases} e^{-\frac{1}{t^2}}, & t > 0, \\ 0, & t \le 0, \end{cases}$$
then $u + \tilde{u}$ is a solution to the Cauchy problem (5.1), (5.2).