ANSWER TO THE MIDTERM EXAMINATION

SOLUTION TO THE 1ST QUESTION

(a). A PDE problem is called to be well-posed if it has the following three properties that:

(1) Existence: The problem has a solution;
(2) Uniqueness: There is at most one solution;
(3) Stability: Solution depends continuously on the data given in the problem.

(b). No, the problem is not well-posed. Let $v = \frac{du}{dx}$, then

$$\frac{dv}{dx} + v = 1, \quad x \in (0,1).$$

The general solution to this equation is

$$v(x) = Ce^{-x} + 1,$$

where C is a constant independent of x. However, by the boundary condition of u, we have $v(0) = 1$ and $v(1) = 0$, which imples C satisfies

$$\begin{cases} C + 1 = 1, \\ Ce^{-1} + 1 = 0, \end{cases}$$

which has no solution.

(c). We show the uniqueness and continuous dependence of solutions in two ways: (Maximum principle)

Claim 1 (Continuous dependence). Suppose $u(t,x) \in C^{1,2}((0,T)\times(0,1))$ is a solution to the problem, and $\phi(x)$ is a continous function, then there exists a constant C which only depends on T such that

$$\sup_{(0,T)\times(0,1)} |u| \le C \sup_{(0,1)} |\phi|.$$

Claim 2 (Uniqueness). Suppose $u, v \in C^{1,2}((0,T)\times(0,1))$ are two solutions to the problem, and $\phi(x)$ is a continous function, then $u \equiv v$.

It suffices to prove *Claim* 1.
Let $U(t,x) = e^{-3t-\left(x-\frac{1}{2}\right)^2} u(t,x)$, then

$$\begin{cases} \partial_t U - \partial_x^2 U - 4(x-\frac{1}{2})\partial_x U + \left[1 + 4\left(x-\frac{1}{2}\right)^2\right] U = 0, & (t,x) \in (0,T)\times(0,1), \\ -U(t,0) + \partial_x U(t,0) = 0, \quad U(t,1) + \partial_x U(t,1) = 0, & t \in (0,T), \\ U(0,x) = e^{-\left(x-\frac{1}{2}\right)^2}\phi(x), & x \in [0,1]. \end{cases}$$

Suppose U attains its nonnegative maximum at interior point $(t_0, x_0) \in (0,T)\times(0,1)$, then

$$U(t_0,x_0) \ge 0, \quad \partial_t U(t_0,x_0) = 0, \quad \partial_x U(t_0,x_0) = 0, \quad \partial_x^2 U(t_0,x_0) \le 0.$$

However, by the equation satisfied by U, we find a contradiction, which implies U only attains its nonnegative maximum at $[0,T]\times\{x=0,1\}\cup\{t=0\}\times[0,1]$.

If U attains its nonegative maximum at $(t_1,0)\in[0,T]\times\{x=0\}$, then

$$U(t_1,0)\geq 0, \quad \partial_x U(t_1,0)\leq 0,$$

then by the boundary condition, we find

$$U(t_1,0)=0.$$

If U attains its nonegative maximum at $(t_2,1)\in[0,T]\times\{x=1\}$, then

$$U(t_2,0)\geq 0, \quad \partial_x U(t_2,1)\geq 0,$$

then by the boundary condition, we find

$$U(t_2,1)=0.$$

If U attains its nonegative maximum at $(0,x_1)\in\{t=0\}\times[0,1]$, then

$$U(0,x_1)\leq \max\left\{0, e^{-\left(x-\frac{1}{2}\right)^2}\phi(x)\right\}.$$

Therefore we have

$$\sup_{(0,T)\times(0,1)} U \leq \max\{0, C\sup_{(0,1)}\phi\}.$$

By a similar arguement, we have

$$\sup_{(0,T)\times(0,1)} U \geq \max\{0, C\sup_{(0,1)}-\phi\}.$$

Therefore we have

$$\sup_{(0,T)\times(0,1)} |u| \leq C\sup_{(0,1)}|\phi|.$$

(Energy method)

Claim 3 (Continuous dependence). Suppose $u(t,x)\in C^{1,2}((0,T)\times(0,1))$ is a solution to the problem, and $\phi(x)$ is a continous function, then there exists a constant C which only depends on T such that

$$\sup_{0\leq t\leq T}\int_0^1 |u(t,x)|^2dx + \int_0^T\int_0^1 |u_x(t,x)|^2dxdt \leq C\int_0^1 |\phi(t,x)|^2dx.$$

Claim 4 (Uniqueness). Suppose $u,v\in C^{1,2}((0,T)\times(0,1))$ are two solutions to the problem, and $\phi(x)$ is a continous function, then $u\equiv v$.

It suffices to prove *Claim* 3.

Multiplying u to both sides of the equation and integrating the resultant with respect to (t,x) over $(0,T)\times(0,1)$, we have

$$\int_0^1 \frac{1}{2}|u(t,x)|^2dx - \int_0^T\int_0^1 (u(t,x)u_x(t,x))_x - |u_x(t,x)|^2dxdt = \int_0^1 \frac{1}{2}|u(0,x)|^2dx.$$

Then by the initial condition and boundary condition, we have

$$\int_0^1 \frac{1}{2}|u(t,x)|^2dx + \int_0^T\int_0^1 |u_x(t,x)|^2dxdt = \int_0^1 \frac{1}{2}|\phi(x)|^2dx.$$

SOLUTION TO THE 2ND QUESTION

(a). For arbitrary $B_\rho(x) \subset \Omega$, denote $n(x)$ to be the outward normal vector at $x \in \partial B_\rho(x)$, then we have

$$\begin{aligned}\int_{B_\rho(x)} \Delta v(y)dy =& \int_{\partial B_\rho(x)} \nabla v(y) \cdot n(y) dS_y \\ =& \rho^n \int_{|w|=1} \nabla v(x+\rho w) \cdot w dw \\ =& \rho^n \int_{|w|=1} \frac{\partial v(x+\rho w)}{\partial \rho} dw \\ =& \rho^n \frac{\partial}{\partial \rho} \int_{|w|=1} v(x+\rho w) dw,\end{aligned}$$

which implies

$$\frac{\partial}{\partial \rho} \int_{|w|=1} v(x+\rho w)dw \geq 0,$$

integrating the above inequality from 0 to r, we have

$$\int_{|w|=1} v(x)dw \leq \int_{|w|=1} v(x+rw)dw,$$

therefore

$$v(x) \leq \frac{3}{4\pi r^3} \int_{B_r(x)} v(y)dy.$$

(b). Denote $M = \max_{\bar{\Omega}} v(x)$, and define $\Omega_M = \{x \in \Omega : v(x) = M\}$. Then since for arbitrary $x \in \Omega_M$,

$$v(x) \leq \frac{3}{4\pi r^3} \int_{B_r(x)} v(y)dy, \quad \forall B_r(x) \subset \Omega,$$

which implies x is a interior point of Ω_M, therefore Ω_M is open, since u is continous, Ω_M is also relatively closed in Ω. Suppose v is not constant and it attains its maximum value only in Ω, then Ω_M is not empty, therefore $\Omega_M = \Omega$ which means v is constant, a contradiction! Therefore

$$\max_{\bar{\Omega}} v(x) = \max_{\partial\Omega} v(x).$$

SOLUTION TO THE 3RD QUESTION

(a). By direct computation,

$$\begin{aligned}(\partial_t - \partial_x^2) \log h =& \frac{\partial_t h}{h} - \frac{\partial_x^2 h}{h} + \frac{|\partial_x h|^2}{h^2} \\ =& \frac{|\partial_x h|^2}{h^2},\end{aligned}$$

$$\begin{aligned}(\partial_t - \partial_x^2) h \log h =& \log h \partial_t h + \partial_t h - \log h \partial_x^2 h - \partial_x^2 h - \frac{|\partial_x h|^2}{h} \\ =& -\frac{|\partial_x h|^2}{h},\end{aligned}$$

$$
\begin{aligned}
(\partial_t-\partial_x^2)\frac{|\partial_x h|^2}{h}=&-\frac{|\partial_x h|^2\partial_t h}{h^2}+\frac{2\partial_x h\partial_{xt}h}{h}-\partial_x(-\frac{|\partial_x h|^3}{h^2}+\frac{2\partial_x h\partial_x^2 h}{h})\\
=&-\frac{|\partial_x h|^2\partial_t h}{h^2}+\frac{2\partial_x h\partial_{xt}h}{h}\\
&-\left(\frac{2|\partial_x h|^4}{h^3}-\frac{3|\partial_x h|^2\partial_x^2 h}{h^2}-\frac{2|\partial_x h|^2\partial_x^2 h}{h^2}+\frac{2|\partial_x^2 h|^2}{h}+\frac{2\partial_x h\partial_x^3 h}{h}\right)\\
=&-2h\left|\frac{\partial_x^2 h}{h}-\frac{|\partial_x h|^2}{h^2}\right|^2.
\end{aligned}
$$

(b). By direct computation,

$$
\begin{aligned}
\frac{d}{dt}H(t)=&\int_0^{2\pi}\partial_t[h(t,x)\log(h(t,x))]dx\\
=&\int_0^{2\pi}\partial_x[(\log(h(t,x))+1)\partial h(t,x)]-\frac{|\partial_x h(t,x)|^2}{h(t,x)}dx\\
=&-\int_0^{2\pi}\frac{|\partial_x h(t,x)|^2}{h(t,x)}dx\\
\leq&0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2}{dt^2}H(t)=&\int_0^{2\pi}\partial_t^2[h(t,x)\log(h(t,x))]dx\\
=&\int_0^{2\pi}\partial_{tx}[(\log(h(t,x))+1)\partial h(t,x)]-\partial_t\left[\frac{|\partial_x h(t,x)|^2}{h(t,x)}\right]dx\\
=&\int_0^{2\pi}-\partial_x^2\left[\frac{|\partial_x h(t,x)|^2}{h(t,x)}\right]+2h\left|\frac{\partial_x^2 h}{h}-\frac{|\partial_x h|^2}{h^2}\right|^2dx\\
=&\int_0^{2\pi}2h\left|\frac{\partial_x^2 h}{h}-\frac{|\partial_x h|^2}{h^2}\right|^2dx\\
\geq&0.
\end{aligned}
$$

(c).

(i). Integrate both sides of the equation with respect to x over $[0,2\pi]$, we have

$$\frac{d}{dt}\int_0^{2\pi}h(t,x)dx=0,$$

which implies

$$\int_0^{2\pi}h(t,x)dx=\int_0^{2\pi}h_0(x)dx=1.$$

(ii). By direct computation,

$$
\begin{aligned}
\frac{dF}{dt}=&\int_0^{2\pi}\partial_t\left[\frac{|\partial_x h(t,x)|^2}{h(t,x)}\right]dx\\
=&\int_0^{2\pi}\partial_x^2\left[\frac{|\partial_x h(t,x)|^2}{h(t,x)}\right]-2h\left|\frac{\partial_x^2 h}{h}-\frac{|\partial_x h|^2}{h^2}\right|^2dx\\
=&-\int_0^{2\pi}2h\left|\frac{\partial_x^2 h}{h}-\frac{|\partial_x h|^2}{h^2}\right|^2dx,
\end{aligned}
$$

therefore

$$\frac{dF}{dt} + 2J = 0.$$

(iii). By direct computation,

$$\begin{aligned} A(\lambda) =& \int_0^{2\pi} h(t,x) \left(\left| \frac{\partial_x^2 h(t,x)}{h(t,x)} - \frac{|\partial_x h(t,x)|^2}{h(t,x)} \right|^2 - 2\lambda \left(\frac{\partial_x^2 h(t,x)}{h(t,x)} - \frac{|\partial_x h(t,x)|^2}{h(t,x)} \right) + \lambda^2 \right) dx \\ =& J - 2\lambda F + \lambda^2 \\ =& (\lambda - F)^2 + J - F^2, \end{aligned}$$

since $A(\lambda) \geq 0$, therefore let $\lambda = F$, we have

$$J - F^2 \geq 0.$$

(d). By direct computation,

$$\begin{aligned} \frac{d^2 N(t)}{dt^2} =& -2\exp(-2H(t))\frac{d^2 H(t)}{dt^2} + 4\left|\frac{dH(t)}{dt}\right|^2 \exp(-2H(t)) \\ =& 4\exp(-2H(t)) \left(-\int_0^{2\pi} h \left| \frac{\partial_x^2 h}{h} - \frac{|\partial_x h|^2}{h^2} \right|^2 dx + \left| \int_0^{2\pi} \frac{|\partial_x h(t,x)|^2}{h(t,x)} dx \right|^2 \right) \\ =& 4\exp(-2H(t))(J - F^2) \\ \geq& 0. \end{aligned}$$

(e). Since

$$\frac{d}{dt}\left(\frac{1}{u}\right) \leq -K,$$

then integrate the above inequality with respect to t over $[0, T]$ for arbitrary $T \geq 0$, we have

$$\frac{1}{u(T)} - \frac{1}{u(0)} \leq -KT,$$

which implies

$$u(T) \leq \frac{u(0)}{1 + Ku(0)T}, \quad \forall T \geq 0.$$

(f). Since

$$\frac{dF}{dt} + 2J = 0,$$

and $J \geq F^2$, then

$$\frac{dF}{dt} + 2F^2 \leq 0,$$

therefore

$$F(t) \leq \frac{F(0)}{1 + 2F(0)t}, \quad \forall T \geq 0.$$

For $t \geq 1$, let $C = \frac{1}{2}$, then we have

$$F(t) \leq \frac{C}{t}, \quad \forall t \geq 1,$$

which implies

$$\int_0^{2\pi} \frac{|\partial_x h(t,x)|^2}{h(t,x)} dx \leq \frac{C}{t}, \quad \forall t \geq 1.$$