ANSWER TO HOMEWORK IV

Solution 1. (i) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx$.

(ii) $\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{n^2\pi}[(-1)^n-1]\cos nx$.

(iii) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx$.

Solution 2. $u(t,x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2}{n}\sin nx\cos nt$.

Solution 3. (i) Since the Fourier series of $f(x)=|x|$ is

$$\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{n^2\pi}[(-1)^n-1]\cos nx,$$

then by Parseval's identity,

$$\frac{\pi^3}{2}+\sum_{n=0}^{\infty}\frac{16}{(2n+1)^4\pi}=\int_{-\pi}^{\pi}|x|^2dx=\frac{2\pi^3}{3},$$

therefore

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}=\frac{\pi^4}{96}.$$

Denote

$$S:=\sum_{n=1}^{\infty}\frac{1}{n^4},\quad S_1:=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4},\quad S_2:=\sum_{n=1}^{\infty}\frac{1}{(2n)^4},$$

then

$$\begin{aligned}S=&S_1+S_2,\\ S_2=&\frac{1}{16}S,\end{aligned}$$

therefore

$$S=\frac{\pi^4}{90}.$$

(ii) Since the Fourier series of $f(x)=x(\pi-x)$ is

$$\sum_{n=1}^{\infty}\frac{4}{n^3\pi}[1-(-1)^n]\sin nx,$$

then by Parseval's identity,

$$\sum_{n=0}^{\infty}\frac{64}{(2n+1)^6\pi}=\int_{-\pi}^{0}|x(\pi+x)|^2dx+\int_{0}^{\pi}|x(\pi-x)|^2dx=\frac{\pi^5}{15},$$

therefore

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6}=\frac{\pi^6}{960}.$$

Denote

$$S := \sum_{n=1}^{\infty} \frac{1}{n^6}, \quad S_1 = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^6}, \quad S_2 := \sum_{n=1}^{\infty} \frac{1}{(2n)^6},$$

then

$$S = S_1 + S_2,$$
$$S_2 = \frac{1}{64} S,$$

therefore

$$S = \frac{\pi^6}{945}.$$

Solution 4. 1. By direct computation,

$$\|f_1\|^2_{H^1(\mathbb{T}^2)} = 4\pi^2 |C|^2,$$
$$\|f_2\|^2_{H^1(\mathbb{T}^2)} = 2\pi^2 (1+n^2),$$
$$\|f_3\|^2_{H^1(\mathbb{T}^2)} = 2\pi^2 (1+n^2).$$

2.
(a) By direct computation,

$$\text{div} U_{n,\omega} = 0,$$
$$\partial_t U_{n,\omega} + (U_{n,\omega} \cdot \nabla_x) U_{n,\omega} = -\frac{1}{n} \begin{pmatrix} \cos(nx_1 - \omega t)\sin(nx_2 - \omega t) \\ \cos(nx_2 - \omega t)\sin(nx_1 - \omega t) \end{pmatrix},$$
$$\text{div}\left((U_{n,\omega} \cdot \nabla_x) U_{n,\omega} \right) = 2\sin(nx_1 - \omega t)\sin(nx_2 - \omega t).$$

(b) By direct computation,

$$\begin{aligned} &\Delta\left(\frac{1}{2n^2}\omega_n\right) \\ =& \frac{\partial\left(\frac{1}{2n}\cos(nx_1 - \omega t)\sin(nx_2 - \omega t)\right)}{\partial x_1} + \frac{\partial\left(\frac{1}{2n}\cos(nx_2 - \omega t)\sin(nx_1 - \omega t)\right)}{\partial x_2} \\ =& -\sin(nx_1 - \omega t)\sin(nx_2 - \omega t) \\ =& -\omega_n. \end{aligned}$$

(c) By direct computation and the above result,

$$\Delta R = \Delta Q + \Delta\left(\frac{1}{2n^2}\omega_n\right) = 0.$$

(d) By the above result,

$$\Delta R = 0,$$

then

$$-|k|^2 \hat{R}(k) = 0, \quad \forall k \in \mathbb{Z}^2,$$

which implies

$$\hat{R}(k) = 0, \quad \forall k \geq 1,$$

therefore

$$R(x) = \frac{1}{(2\pi)^2}\hat{R}(0).$$

(e) By direct computation and the above results,

$$\begin{aligned}
&\nabla_x \Delta^{-1} \mathrm{div}((U_{n,\omega} \cdot \nabla_x) U_{n,\omega}) \\
=&\nabla_x \Delta^{-1}(2\omega_n) \\
=&\nabla_x \left(-\frac{1}{n^2}\omega_n + 2C_n \right) \\
=&-\frac{1}{n} \begin{pmatrix} \cos(nx_1 - \omega t)\sin(nx_2 - \omega t) \\ \cos(nx_2 - \omega t)\sin(nx_1 - \omega t) \end{pmatrix}.
\end{aligned}$$

(f) By direct computation and the above results,

$$\partial_t U_{n,\omega} + (U_{n,\omega} \cdot \nabla_x) U_{n,\omega} - \nabla_x \Delta^{-1} \mathrm{div}\left((U_{n,\omega} \cdot \nabla_x) U_{n,\omega} \right) = 0.$$

3. Since

$$\|U_n^0 - V_n^0\|_{H^1(\mathbb{T}^2)}^2 = \frac{8\pi^2}{n^2},$$

therefore

$$\lim_{n\to\infty} \|U_n^0 - V_n^0\|_{H^1(\mathbb{T}^2)} = 0.$$

4. By the previous results, we have

$$U_n = \begin{pmatrix} \frac{1+\cos(nx_2 - t)}{n} \\ \frac{1+\cos(nx_1 - t)}{n} \end{pmatrix}, \quad V_n = \begin{pmatrix} \frac{-1+\cos(nx_2 + t)}{n} \\ \frac{-1+\cos(nx_1 + t)}{n} \end{pmatrix},$$

therefore

$$\begin{aligned}
\|U_n - V_n\|_{H^1(\mathbb{T}^2)}^2 =& \frac{8\pi^2[5 - \cos(2t)]}{n^2} + 16\pi^2(\sin t)^2 \\
\geq& \frac{32\pi^2}{n^2} + 16\pi^2(\sin t)^2,
\end{aligned}$$

which implies

$$\|U_n - V_n\|_{H^1(\mathbb{T}^2)} \geq 2\sqrt{2}\pi^2 |\sin t| - \frac{4\pi}{n}.$$