ANSWER TO HOMEWORK II

Solution 1. Recall the fundamental solution to the Laplace equation $\Delta u = 0$ in $\mathbb{R}^2$,

$$\Phi(x,y) = -\frac{1}{4\pi}\log(x^2+y^2).$$

Therefore we construct the Green's function in the following ways.

(1)

$$G(x,y;\xi,\eta) = -\frac{1}{4\pi}\log\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}.$$

(2)

$$G(x,y;\xi,\eta) = -\frac{1}{4\pi}\log\left[\frac{(x-\xi)^2+(y-\eta)^2}{(x+\xi)^2+(y-\eta)^2}\frac{(x+\xi)^2+(y+\eta)^2}{(x-\xi)^2+(y+\eta)^2}\right].$$

Solution 2. Recall the fundamental solution to the heat equation $u_t - \Delta u = 0$ in $\mathbb{R}_+ \times \mathbb{R}^n$,

$$\Phi(t,x) = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}.$$

Therefore we construct the Green's function in the following ways.

(1)

$$G(t,x;\tau,\xi) = \frac{1}{\sqrt{4\pi(t-\tau)}}\sum_{n=-\infty}^{\infty}\left[e^{-\frac{|x-\xi-2nl|^2}{4(t-\tau)}} - e^{-\frac{|x+\xi-2nl|^2}{4(t-\tau)}}\right].$$

(2)

$$G(t,x,y;\tau,\xi,\eta) = \frac{1}{4\pi(t-\tau)}\left[e^{-\frac{|x-\xi|^2+|y-\eta|^2}{4(t-\tau)}} - e^{-\frac{|x-\xi|^2+|y+\eta|^2}{4(t-\tau)}} + e^{-\frac{|x+\xi|^2+|y-\eta|^2}{4(t-\tau)}} - e^{-\frac{|x+\xi|^2+|y+\eta|^2}{4(t-\tau)}}\right].$$

Solution 3. (1) Let $v(t,x) = e^{-t}u(t,x)$, then

$$v_t(t,x) - v_{xx}(t,x) = 0,$$

therefore

$$v(t,x) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}}\phi(y)dy,$$

then

$$u(t,x) = e^t\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}}\phi(y)dy.$$

(2) Let $v(t,x) = u(t,x-t)$, then

$$v_t(t,x) - v_{xx}(t,x) = 0,$$

therefore

$$v(t,x) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}}\phi(y)dy,$$

then

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{|x+t-y|^2}{4t}} \phi(y) dy.$$

Solution 4. (1) Since for $(t,x) \in \{t=0\} \times [0,1]$,

$$u(t,x) = x(1-x) \geq 0,$$

and for $(t,x) \in [0,\infty) \times \{x = 0,1\}$,

$$u(t,x) = 0,$$

therefore by the maximum principle, we have

$$u(t,x) \geq 0, \quad \forall (t,x) \in [0,\infty) \times [0,1].$$

(2) Denote $w(t,x) = x(1-x)e^{-t}$, and let $v = w - u$, then

$$\begin{aligned} v_t(t,x) - v_{xx}(t,x) &= (2 - x(1-x))e^{-t}, \quad (t,x) \in (0,\infty) \times [0,1], \\ v(0,x) &= 0, \quad x \in [0,1], \\ v(t,0) &= v(t,1) = 0, \quad t \in (0,\infty). \end{aligned}$$

Since for $(t,x) \in \{t=0\} \times [0,1] \cup [0,\infty) \times \{x = 0,1\}$,

$$v(t,x) = 0,$$

and

$$v_t(t,x) - v_{xx}(t,x) \geq 0,$$

then by the maximum principle, we have

$$v(t,x) \geq 0,$$

which implies

$$u(t,x) \leq x(1-x)e^{-t}, \quad \forall (t,x) \in [0,\infty) \times [0,1].$$

Since $u(t,x) \geq 0$, therefore

$$\lim_{t\to\infty} \sup_{x\in[0,1]} |u(t,x)| \leq \lim_{t\to\infty} \sup_{x\in[0,1]} |x(1-x)e^{-t}| = 0.$$

Solution 5. (1) We first assume $8c(T+1) < 1$. For arbitrary $y \in \mathbb{R}$, define

$$v(t,x) = u(t,x) - \frac{\varepsilon}{\sqrt{T+1-t}} e^{\frac{|x-y|^2}{4(T+1-t)}}, \quad (t,x) \in [0,T] \times \mathbb{R}.$$

Then

$$v_t(t,x) - v_{xx}(t,x) = 0$$

Let $r > 0$ and consider $[0,T] \times [y-r, y+r]$, if $(t,x) \in \{t=0\} \times [y-r,y+r]$, then

$$\begin{aligned} v(0,x) =& u(0,x) - \frac{\varepsilon}{\sqrt{T+1}} e^{\frac{|x-y|^2}{4(T+1)}} \\ \leq& \sup_{\mathbb{R}} \phi(x), \end{aligned}$$

if $(t,x) \in [0,T] \times \{x = y-r, y+r\}$, then

$$\begin{aligned} v(t,x) =& u(t,x) - \frac{\varepsilon}{\sqrt{T+1-t}} e^{\frac{|x-y|^2}{4(T+1-t)}} \\ \leq& Ce^{c(|y|+r)^2} - \frac{\varepsilon}{\sqrt{T+1}} e^{\frac{r^2}{4(T+1)}} \\ \leq& Ce^{c(|y|+r)^2} - \frac{\varepsilon}{\sqrt{T+1}} e^{2cr^2}, \end{aligned}$$

by choosing r sufficiently large,

$$v(t,x) \leq \sup_{\mathbb{R}} \phi(x).$$

Therefore by the maximum principle, we have

$$\sup_{(0,T)\times[y-r,y+r]} v \leq \sup_{\mathbb{R}} \phi(x),$$

Since y is arbitrary,

$$\sup_{(0,T)\times\mathbb{R}} v \leq \sup_{\mathbb{R}} \phi(x),$$

then by letting ε goes to 0, we have

$$\sup_{(0,T)\times\mathbb{R}} u \leq \sup_{\mathbb{R}} \phi(x).$$

In general, for arbitrary $T > 0$, we repeatedly apply the above result on the time intervals $[0, T_1]$, $[T_1, 2T_1]$, ..., where $T_1 > 0$ such that $8c(T_1 + 1) \leq 1$, then we have

$$\sup_{(0,T)\times\mathbb{R}} u \leq \sup_{\mathbb{R}} \phi(x).$$

Apply the above result for $-u$, we also have

$$\sup_{(0,T)\times\mathbb{R}} -u \leq \sup_{\mathbb{R}} -\phi(x).$$

therefore

$$\sup_{(0,T)} |u| \leq \sup_{\mathbb{R}} |\phi(x)|.$$

(2) It suffices to show that

$$\tilde{u}(t,x) = \sum_{k=0}^{\infty} \frac{d^k\varphi(t)}{dt^k} \frac{x^{2k}}{(2k)!},$$

satisfies

$$\tilde{u}_t(t,x) - \tilde{u}_{xx}(t,x) = 0,$$

and $\tilde{u}(0,x) = 0$.

Firstly, we prove $\tilde{u}$ is well-defined. Since

$$\left|\frac{d^k}{dt^k}\varphi(t)\right| \leq k!\left(\frac{2}{t}\right)^k e^{-\frac{1}{4t^2}}, \quad \forall k \in \mathbb{N},$$

therefore for arbitrary $|x| \leq r$,

$$\begin{aligned} \left|\sum_{k=0}^{\infty} \frac{d^k\varphi(t)}{dt^k} \frac{x^{2k}}{(2k)!}\right| \leq& e^{-\frac{1}{4t^2}} \sum_{k=0}^{\infty} k!\left(\frac{2}{t}\right)^k \frac{r^{2k}}{(2k)!} \\ \leq& e^{-\frac{1}{4t^2}+\frac{r^2}{t}}, \end{aligned}$$

which implies $\tilde{u}$ is well-defined.

Secondly, we prove $\tilde{u}$ satisfies

$$\tilde{u}_t(t,x) - \tilde{u}_{xx}(t,x) = 0,$$

and $\tilde{u}(0,x) = 0$. Indeed, by direct computation, we have $\tilde{u}(0,x) = 0$, and

$$\begin{aligned}
\frac{\partial^2 \tilde{u}}{\partial x^2} &= \sum_{k=0}^{\infty} \frac{d^k \varphi(t)}{dt^k} \frac{\partial^2}{\partial x^2} \left(\frac{x^{2k}}{(2k)!} \right) \\
&= \sum_{k=1}^{\infty} \frac{d^k \varphi(t)}{dt^k} 2k(2k-1) \frac{x^{2k-2}}{(2k)!} \\
&= \sum_{k=1}^{\infty} \frac{d^k \varphi(t)}{dt^k} \frac{x^{2(k-1)}}{(2(k-1))!} \\
&= \sum_{k=0}^{\infty} \frac{d^{k+1} \varphi(t)}{dt^{k+1}} \frac{x^{2k}}{(2k)!} \\
&= \frac{\partial \tilde{u}}{\partial t}.
\end{aligned}$$