MATH4210: Financial Mathematics Tutorial 10

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Options

Proposition (Law of One Price)

If two portfolios have the same profit at maturity time T, then for all prior times $t < T$, the price of the portfolio's must be equal.

Proof.

By no-arbitrage, it is easy to prove by contradictions. □

Question

Show that the European put options with strike price K and maturity at time T satisfies $P_E(t, K) > Ke^{-r(T-t)} - S(t)$ for all $t < T$, where $S(t)$ is the stock price, r is the continuous compounded interest rate.

Suppose $\exists t_0 \in [0, T)$, s.t. $P_E(t_0, K) \leq Ke^{-r(T-t_0)} - S(t_0)$.

Construct Π: long 1 put option, short 1 $(Ke^{-r(T-t_0)} - S(t_0))$.

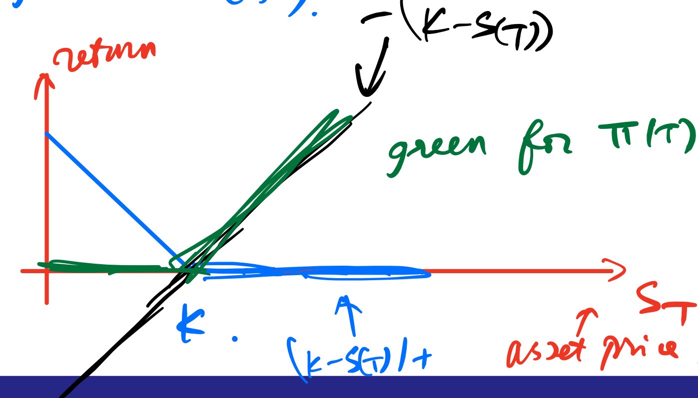
Options

$\pi(T) = P_E(T,K) - K + S(T) = (K-S(T))_+ - (K-S(T)) \quad \cdots \ (1).$

$\pi(t_0) = \cdots$

Goal: $\begin{cases} \mathbb{P}[\pi(T)>0] > 0 \\ \mathbb{P}[\pi(T) \geq 0] = 1 \end{cases}$

Since $\pi(T) = \begin{cases} \cdots & S_T > K \\ \cdots & S_T \leq K \end{cases}$



Question

Two vanilla put options are identical except for the maturity dates $T_1 < T_2$. If the interest rate is zero between T_1 and T_2, then $P_E(t, T_1) < P_E(t, T_2)$ at any time $t \leq T_1$.

Suppose $P_E(t,T_1) \geq P_E(t,T_2)$

π: Short 1 $P_E(t,T_1)$. Long $P_E(t,T_2)$

$\pi(0) = P_E(t,T_2) - P_E(t,T_1) \leq 0$

$\pi(T_1) = P_E(T_1,T_2) - (K-S(T_1))_+ \quad \cdots \ (1)$

By previous question:

$\forall t<T,\ P_E(t,T) > Ke^{-r(T-t)} - S(t) \quad \cdots \ (2)$

(1), (2) gives.

$\pi(T_1) > K\boxed{\cancel{e^{-r(T_2-T_1)}}} - S(T_1) - (K-S(T_1))_+$

$= K - S(T_1) - (K-S(T_1))_+$

$= 0.$ { either discuss / or draw a graph.

$\Rightarrow \pi(T_1) > 0.$

$\Rightarrow \begin{cases} \mathbb{P}(\pi(T) > 0) > 0 \\ \mathbb{P}(\pi(T) \geq 0) = 1 \end{cases}$

Options

Method 1: (By contradiction)

$\begin{cases} 0 < P_E(t,K_2) - P_E(t,K_1) & (\geq) \rightarrow P_E(t,K_2) \leq P_E(t,K_1) \\ P_E(t,K_2) - P_E(t,K_1) < (K_2 - K_1)e^{-r(T-t)} & (\geq). \end{cases}$

Contradictions.

(Assume $\geq$): $\Pi(t) = (K_2-K_1)e^{-r(T-t)} - P_E(t,K_2) + P_E(t,K_1)$, $\Pi(T) = (K_2-K_1) - (K_2-S_T)_+ + (K_1-S_T)_+$

Question

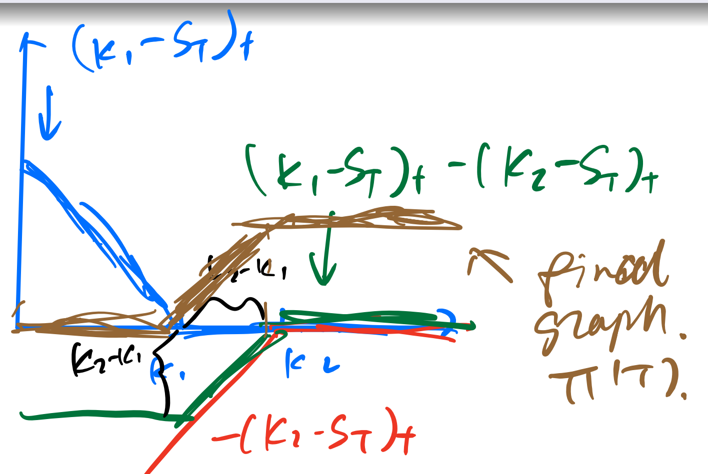
Suppose two put European options are identical except for the strike prices $0 < K_1 < K_2$, *show that*

$0 < C_E(t,K_1) - C_E(t,K_2) < (K_2-K_1)e^{-r(T-t)}$

$$0 < P_E(t,K_2) - P_E(t,K_1) < (K_2-K_1)e^{-r(T-t)},$$

at any time t before maturity T.

$\boxed{CK} = \boxed{PS} \Longleftrightarrow C_E + Ke^{-r(T-t)} = P_E + S_T$

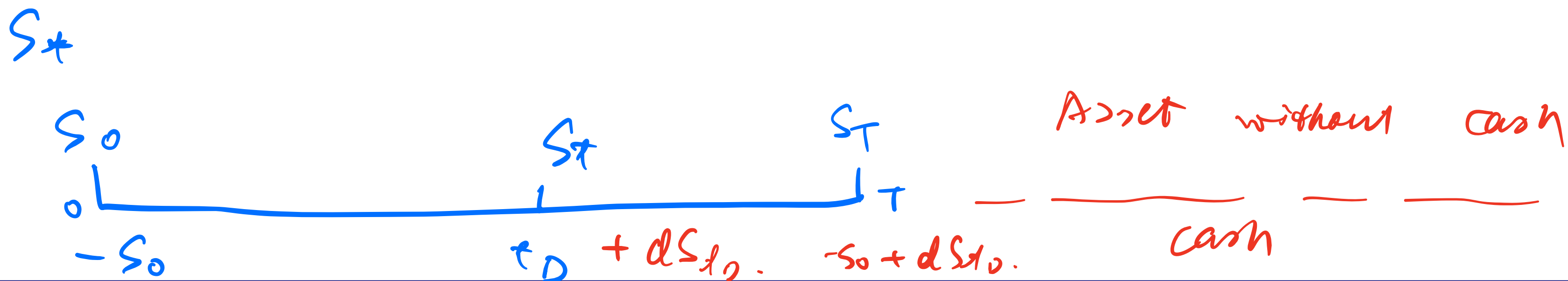


Method 2. (put call parity)

$P_E(t,K_2) - P_E(t,K_1)$

$= C_E(t,K_2) - C_E(t,K_1) + K_2e^{-r(T-t)} - K_1e^{-r(T-t)}$

Options



Question (Put-Call Parity Relation with Dividend)

Prove the following. Assume that the value of the dividends of the stock paid during $[t, T]$ is a deterministic constant D at time $t_D \in (t, T]$. Let $S(t)$ be the stock price, r be the continuous compounding interest rate, $C_E(t, K)$ and $P_E(t, K)$ be the prices of European call and put option at time t with strike K and maturity T respectively. We have

$$\boxed{C_E(t, K) - P_E(t, K)} = \boxed{S(t) - Ke^{-r(T-t)} - De^{-r(t_D-t)}}$$

Π_1: long $C_E(t,K)$, short $P_E(t,K)$, $\Pi_1(t) = c$

$\Pi_1(T) = (S_T-K)_+ - (K-S_T)_+ = S_T - K$.

Π_2: long $S(t)$, short $(Ke^{-r(T-t)} + De^{-r(t_D-t)})$, at time t_D, we use dividend to repay the debt of $De^{-r(t_D-t)}$,

$\Pi_2(t_D) = S(t_D) - Ke^{-r(T-t_D)} - D + D = S(t_D) - Ke^{-r(T-t_D)}$

Forward

$\Pi_2(T) = S(T) - K = \Pi_1(T).$

$\Rightarrow \Pi_1(t) = \Pi_2(t) \quad \forall t \in [0, T]$
by law of one price

Question

Under no arbitrage opportunity assumptions and assume the continuous compounded interest rate is r, if the stock pays no dividend, show that $F(t, T) = S(t)e^{r(T-t)}$ for $t \geq T$.

Π_1: long $\boxed{F(t,T)}$, put $\boxed{\$ F(t,T)e^{-r(T-t)}}$ in the bank

Π_2: long $S(t)$.

$\Pi_1(t) = F(t,T)e^{-r(T-t)}$.

$\Pi_1(T) = \underset{\text{Money from bank}}{F(t,T)} + (\underset{\text{get 1 } S(T)}{S(T)} - \underset{\text{pay predetermined price}}{F(t,T)}) = S(T) = \Pi_2(T)$

$\Pi_1(t) = \Pi_2(t) \Rightarrow S(t) = F(t,T)e^{-r(\cdot)}$

Forward

Question

Suppose the stock pay a dividend $d \times S(t)$ at time t, where $0 < t < T$ and $0 < d < 1$, show its forward price $F(0,T) = \dfrac{1}{1+d} S(0) e^{rT}$.

Π_1 : long $S(0)$ at initial time

$\Pi_1(t) = S(t) + d\,S(t) = (1+d)S(t)$.

$\Pi_1(T) = (1+d)S(T)$.

Π_2 : long $(1+d)F(0,T)$, put $(1+d)F(0,T)e^{-rT}$ in the bank.

$\Pi_2(0) = (1+d)F(0,T)e^{-rT}$

$\Pi_2(T) = \cancel{(1+d)F(0,T)} + \left((1+d)S(T) - \cancel{(1+d)F(0,T)}\right) = (1+d)S(T) = \Pi_1(T)$

So $\Pi_1(0) = \Pi_2(0)$

$\Rightarrow S(0) = (1+d)F(0,T)e^{-rT}$

$\Rightarrow F(0,T) = \frac{1}{1+d} S(0) e^{rT}$