THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4060 Complex Analysis 2022-23
Tutorial 9
30th March 2022

1. (Ex.15 Ch.8 in textbook) Here are two properties enjoyed by automorphisms of the upper half plane.

   (a) Suppose Φ is an automorphism of $\mathbb{H}$ that fixes three distinct points on the real axis. Then Φ is the identity.

   (b) Suppose (x_1, x_2, x_3) and (y_1, y_2, y_3) are two pairs of three distinct points on the real axis with

   $$x_1 < x_2 < x_3 \text{ and } y_1 < y_2 < y_3$$

   Prove that there exists (a unique) automorphism Φ of $\mathbb{H}$ so that $\Phi(x_j) = y_j, j = 1, 2, 3$. The same conclusion holds if $y_3 < y_1 < y_2$ or $y_2 < y_3 < y_1$.

Solution. (a) The fixed points of the automorphism is given by the equation:

$$z = \frac{az+b}{cz+d} \quad \Rightarrow cz^2 + (d-a)z - b = 0$$

The only possible case for this equation having 3 distinct solution is $c = d - a = b = 0$. Thus the automorphism must be identity.

(b) For uniqueness, if we have two function φ_1 and φ_2 satisfying the condition, we have $\varphi_1 \circ \varphi_2^{-1}$ fixes y_1, y_2, y_3. Thus by part (a), $\varphi_1 \circ \varphi_2^{-1} = Id \Rightarrow \varphi_1 = \varphi_2$.
For the existence, Consider the function defined by the equation:

$$\frac{(z-x_2)(x_1-x_3)}{(z-x_3)(x_1-x_2)} = \frac{(\Phi(z)-y_2)(y_1-y_3)}{(\Phi(z)-y_3)(y_1-y_2)}$$

It is also a linear fractional transformation:

$$\Phi(z) = \frac{(y_3 - y_2 t)z - y_3 x_2 + x_3 y_2 t}{(1-t)z - x_2 + x_3 t} \quad \text{where} \quad t = \frac{(y_1-y_3)(x_1-x_2)}{(y_1-y_2)(x_1-x_3)}$$

Easy to check the function maps (x_1, x_2, x_3) to (y_1, y_2, y_3).

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2. (Ex.16 Ch.8 in textbook) Let

$$f(z) = \frac{i-z}{i+z} \text{ and } f^{-1}(w) = i\frac{1-w}{1+w}$$

   (a) Given $\theta \in \mathbb{R}$, find real numbers a, b, c, d so that $ad - bc = 1$, and so that for any $z \in \mathbb{H}$,

   $$\frac{az+b}{cz+d} = f^{-1}\left(e^{i\theta} f(z)\right)$$

(b) Given $\alpha \in \mathbb{D}$, find real numbers a, b, c, d such that $ad - bc = 1$, and so that for any $z \in \mathbb{H}$,

$$\frac{az+b}{cz+d} = f^{-1}\left(\psi_\alpha(f(z))\right)$$

(c) Prove that if g is an automorphism of the unit disc, then there exist real numbers a, b, c, d such that $ad - bc = 1$ and so that for any $z \in \mathbb{H}$,

$$\frac{az+b}{cz+d} = f^{-1} \circ g \circ f(z).$$

Solution. (a) Since $f,\ f^{-1}, e^{i\theta}z$ are all linear fractional transformation, we can use the matrix product formula:

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}$$
$$= 2ie^{i\theta/2}\begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

so $a = d = \cos\theta/2, b = -c = \sin\theta/2$.

(b) Since $f,\ f^{-1}, \frac{\alpha - z}{1-\bar{\alpha}z}$ are all linear fractional transformation, we can use the matrix product formula:

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & \alpha \\ -\bar{\alpha} & 1 \end{pmatrix}\begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}$$
$$= 2\begin{pmatrix} Im\alpha & Re\alpha - 1 \\ Re\alpha + 1 & -Im\alpha \end{pmatrix}$$

the determinant is $1 - |\alpha|^2$. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sqrt{\frac{1}{1-|\alpha|^2}}\begin{pmatrix} Im\alpha & Re\alpha - 1 \\ Re\alpha + 1 & -Im\alpha \end{pmatrix}$$

(c) By product rule of linear fractional transformation, the result is just the product of the matrices in (a) and (b).

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3. (Ex.19 Ch.8 in textbook) Prove that the complex plane slit along the union of the rays $\cup_{k=1}^{n}\{A_k + iy : y \leq 0\}$ is simply connected.

Solution. Recall that the definition of simply connected region Ω is for any two curves:$\gamma_1$andγ_2 inside the region starting at the same point and ending at the same point, then γ_1 is homotopic to γ_2. (i.e exist a continuous map $F(s,t)$ from $[0,1]\times[0,1]$ to Ω s.t $F(0,t) = \gamma_1(t)$, $F(1,t) = \gamma_2(t)$, $F(s,0) = \gamma_1(0) = \gamma_2(0)$ and $F(s,1) = \gamma_1(1) = \gamma_2(1)$).

Now If we have two curves $\gamma_1(t)$and$\gamma_2(t)$ as above in $\Omega = \mathbb{C} \setminus \cup_{k=1}^{n}\{A_k + iy : y \leq 0\}$. Let $M = \max(\max|Im\gamma_1(t)|, \max|Im\gamma_2(t)|)$. Then let $\tilde{\gamma}_1(t) := \gamma_1(t) + iM$ and $\tilde{\gamma}_2(t) := \gamma_1(t) + iM$. Then $\tilde{\gamma}_1(t)$ and $\tilde{\gamma}_2(t)$ are completely contained in $\mathbb{H}$. Thus there exists a homotopy $\phi(s,t)$ from $\tilde{\gamma}_1(t)$ to $\tilde{\gamma}_2(t)$, Then $\phi(s,t)$ can be directly extended to a homotopy between $\gamma_1(t)$and$\gamma_2(t)$.

It seems the homotopy functions that I have written in tutorial is not completely correct. If you are interested in it, you can try to give the explicit function by yourself :D. ◀

4. (Additional Exercise)Prove the Caratheodory inequality: Let$f : D(0 : R) \to \mathbb{C}$ is a holomorphic function, Suppose $r < R$, then we have:

$$\|f\|_r \leq \frac{2r}{R-r} \sup_{|z|\leq R} Ref(z) + \frac{R+r}{R-r}|f(0)|$$

where

$$\|f\|_r = \max_{|z|\leq r} |f(z)| = \max_{|z|=r} |f(z)|$$

Solution. See wiki: https://en.wikipedia.org/wiki/Borel–Caratheodory_theorem

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