THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4060 Complex Analysis 2022-23
Tutorial 12
20th April 2022

1. (Ex.6 Ch.9 in textbook) Prove that $\wp''$ is a quadratic polynomial in $\wp$.

   Solution. Recall we have the formula:

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

   where $e_1 = \wp(\frac{1}{2}),\ e_2 = \wp(\frac{\tau}{2}),\ e_3 = \wp(\frac{1+\tau}{2})$ are constants. Then we differentiate it on both sides,

$$2\wp'\wp'' = 4\wp'(3\wp^2 - 2(e_1+e_2+e_3)\wp + e_1e_2 + e_2e_3 + e_1e_3)$$

   Thus $\wp'' = 6\wp^2 - 4(e_1+e_2+e_3)\wp + 2e_1e_2 + 2e_2e_3 + 2e_1e_3$ is a quadratic polynomial in $\wp$.

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2. (Ex.7 Ch.9 in textbook) Setting $\tau = 1/2$ in the expression

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+\tau)^2} = \frac{\pi^2}{\sin^2(\pi\tau)}$$

   deduce that

$$\sum_{m\geq 1, m \text{ odd}} \frac{1}{m^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{m\geq 1} \frac{1}{m^2} = \frac{\pi^2}{6} = \zeta(2).$$

   Similarly, using $\sum 1/(m+\tau)^4$ deduce that

$$\sum_{m\geq 1, m \text{ odd}} \frac{1}{m^4} = \frac{\pi^4}{96} \quad \text{and} \quad \sum_{m\geq 1} \frac{1}{m^4} = \frac{\pi^4}{90} = \zeta(4).$$

   These results were already obtained using Fourier series in the exercises at the end of Chapters 2 and 3 in Book I.

   Solution. We have

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^2} = \frac{\pi^2}{\sin^2(\frac{\pi}{2})}$$
$$\Longrightarrow \sum_{m=-\infty}^{\infty} \frac{4}{(2m+1)^2} = \pi^2$$
$$\Longrightarrow \sum_{m=0}^{\infty} \frac{8}{(2m+1)^2} = \pi^2$$
$$\Longrightarrow \sum_{m\geq 1, m \text{ odd}}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{8}$$

For $\sum_{m\geq 1}\frac{1}{m^2}$, because everything is absolutely convergent, we can arbitrarily exchange terms:

$$\begin{aligned}\sum_{m\geq 1}^{\infty}\frac{1}{m^2} &= \sum_{m\geq 1, m\text{ odd}}^{\infty}\frac{1}{m^2}+\sum_{m\geq 2, m\text{ even}}^{\infty}\frac{1}{m^2}\\ &= \sum_{m\geq 1, m\text{ odd}}^{\infty}\frac{1}{m^2}+\sum_{m\geq 1}^{\infty}\frac{1}{4m^2}\\ \Longrightarrow \sum_{m\geq 1}^{\infty}\frac{1}{m^2} &= \frac{4}{3}\sum_{m\geq 1, m\text{ odd}}^{\infty}\frac{1}{m^2}=\frac{\pi^2}{6}\end{aligned}$$

similarly for $\sum_{m=-\infty}^{\infty}\frac{1}{(m+\tau)^4}$, by differentiate

$$\sum_{m=-\infty}^{\infty}\frac{1}{(m+\tau)^2}=\frac{\pi^2}{\sin^2(\pi\tau)}$$

twice and replace τ by $\frac{1}{2}$, we have

$$\sum_{m=-\infty}^{\infty}\frac{1}{(m+\frac{1}{2})^4}=\frac{\pi^4}{3}$$

Then

$$\sum_{m\geq 1, m\text{ odd}}^{\infty}\frac{1}{m^4}=\frac{\pi^4}{96}\quad\text{and}\quad\sum_{m\geq 1}^{\infty}\frac{1}{m^2}=\frac{\pi^4}{90}$$

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3. (Ex.8 Ch.9 in textbook) Let

$$E_4(\tau)=\sum_{(n,m)\neq(0,0)}\frac{1}{(n+m\tau)^4}$$

be the Eisenstein series of order 4 .

(a) Show that $E_4(\tau)\to\pi^4/45$ as $\mathrm{Im}(\tau)\to\infty$.

(b) More precisely,

$$\left|E_4(\tau)-\frac{\pi^4}{45}\right|\leq ce^{-2\pi t}\qquad\text{if }\tau=x+it\text{ and }t\geq 1$$

(c) Deduce that

$$\left|E_4(\tau)-\tau^{-4}\frac{\pi^4}{45}\right|\leq ct^{-4}e^{-2\pi/t}\qquad\text{if }\tau=it\text{ and }0<t\leq 1$$

Solution. (a) It is a consequence of (b).

(b) Recall Theorem 2.5 gives us the formula, for $k \geq 4$ even:

$$E_k(\tau) = 2\zeta(k) + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!}\sum_{r=1}^{\infty}\sigma_{k-1}(r)e^{2\pi i r\tau}$$

Where $\sigma_l(r) = \sum_{d|r} d^l$ From previous result, we know that $\zeta(4) = \frac{\pi^2}{90}$, Thus

$$E_4(\tau) - 2\zeta(4) = E_k(\tau) - \frac{\pi^2}{45} = \frac{2(-1)^2(2\pi)^4}{3!}\sum_{r=1}^{\infty}\sigma_3(r)e^{2\pi i r\tau}$$

We need to estimate the right hand side, in fact,

$$\begin{aligned}
|\frac{2(-1)^2(2\pi)^4}{3!}\sum_{r=1}^{\infty}\sigma_3(r)e^{2\pi i r\tau}| &\leq c_1\sum_{r=1}^{\infty}|\sigma_3(r)e^{2\pi i r\tau}| \\
&\leq c_1\sum_{r=1}^{\infty}r^4e^{-2\pi r t} && \text{where } t = Im\tau \\
&= c_1e^{-2\pi t}\sum_{r=1}^{\infty}r^4e^{-2\pi t(r-1)} \\
&\leq ce^{-2\pi t} && \text{because } e^{-2\pi t(r-1)} \text{ decay rapidly}
\end{aligned}$$

(c) From theorem 2.1(iii), we have

$$E_k(\tau) = \tau^{-k}E_k(-1/\tau)$$

Thus if $\tau = it$ and $0 < t \leq 1$, let $\theta = -1/\tau$, then θ

$$|E_4(\tau) - \tau^{-4}\frac{\pi^4}{45}| = |\theta^4E_4(\theta) - \theta^4\frac{\pi^4}{45}| \leq |\theta^4|ce^{-2\pi/t} = ct^{-4}e^{-2\pi/t}$$

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