THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4060 Complex Analysis 2022-23
Homework 5 solutions

1. (Exercise 17 in textbook) Note that the area of $\mathbb{D}$ is π. After mapping, the area should be

$$\int\int_{\mathbb{D}} |detJ|dxdy = \int\int_{\mathbb{D}} |(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2|dxdy = \int\int_{\mathbb{D}} |\psi'_\alpha|^2 dxdy$$

where J is the Jacobian matrix. On the other hand, the image of $\mathbb{D}$ under ψ'_α is $\mathbb{D}$, thus the area is still π.thus

$$\frac{1}{\pi}\int\int_{\mathbb{D}} |\psi'_\alpha|^2 dxdy = 1$$

For the second integration,

$$\begin{aligned}
\frac{1}{\pi}\int\int_{\mathbb{D}} |\psi'_\alpha|dxdy &= \frac{1-|\alpha|^2}{\pi}\int\int_{\mathbb{D}} \frac{1}{|(1-\overline{\alpha}z)|^2}dxdy \\
&= \frac{1-|\alpha|^2}{\pi}\int_0^1\int_0^{2\pi} \frac{1}{|(1-\overline{\alpha}re^{i\theta})|^2}rd\theta dr \\
\text{(by rotation)} &= \frac{1-|\alpha|^2}{\pi}\int_0^1\int_0^{2\pi} \frac{1}{|(1-|\alpha|re^{i\theta})|^2}rd\theta dr \\
&= \frac{1-|\alpha|^2}{\pi}\int_0^1\int_0^{2\pi} \frac{1}{1+|\alpha|^2r^2-2r|\alpha|\cos\theta}rd\theta dr \\
\text{(by long computation)} &= \frac{1-|\alpha|^2}{\pi}\int_0^1 \frac{2\pi r}{1-|\alpha|^2r^2}dr \\
&= \frac{|\alpha|^2-1}{|\alpha|^2}\ln(1-|\alpha|^2r^2)|_0^1 \\
&= \frac{|\alpha|^2-1}{|\alpha|^2}\ln(1-|\alpha|^2)
\end{aligned}$$

2. (Exercise 18 in textbook) Every thing is the same in the proof of theorem 4.2. But note that if it is not piecewise smooth simple closed curves, the inverse conformal map will not satisfy lemma 4.4. For example, in the case "$\rightarrow | \leftarrow$", the inverse will map the limit point to two different points on the unit circle. Hence cannot use the same way to proof lemma 4.4.

3. (Exercise 20 in textbook)

   (a) By Schwarz-Christoffel formula, since $\beta_1+\beta_2+\beta_3 = 1/2+1/2+1/2 = 3/2 < 2$. It maps the upper half plane to a polygon with four angles $1/2, ..., 1/2$, i.e a rectangle.

   (b) By change of variable $t = \zeta^2$, the side length is

$$|\int_0^1 \frac{d\zeta}{\sqrt{\zeta(\zeta^2-1)}}| = \int_0^1 \frac{d\zeta}{\sqrt{\zeta(1-\zeta^2)}} = \frac{1}{2}\int_0^1 t^{-\frac{3}{4}}(1-t)^{-\frac{1}{2}}dt$$

We can use the formula for Beta function

$$B(\alpha,\beta):=\int_0^1(1-t)^{\alpha-1}t^{\beta-1}dt=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Then

$$\frac{1}{2}\int_0^1 t^{-\frac{3}{4}}(t-1)^{-\frac{1}{2}}dt=\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{2\Gamma(\frac{3}{4})}=\frac{\Gamma^2(\frac{1}{4})\sqrt{\pi}\sin\frac{\pi}{4}}{2\pi}=\frac{\Gamma^2(\frac{1}{4})}{2\sqrt{2\pi}}$$

Similarly, for adjacent edge, by change of variable $t=-\zeta$

$$|\int_{-1}^{0}\frac{d\zeta}{\sqrt{\zeta(1-\zeta^2)}}|=|-i\int_1^0\frac{-dt}{\sqrt{t(1-t^2)}}|=\frac{1}{2}\int_0^1 t^{-\frac{3}{4}}(1-t)^{-\frac{1}{2}}dt$$

Hence it is a square with side length $\frac{\Gamma^2(\frac{1}{4})}{2\sqrt{2\pi}}$.

4. (Exercise 23 in textbook) We can write the integration as

$$F(z)=\frac{1}{(-1)^{2/n}}\int_1^z\frac{d\zeta}{(\zeta-a_1)^{2/n}...(\zeta-a_n)^{2/n}}$$

where a_i are the n-th roots of unit. Then by change of variables (say, pull it back to $\mathbb{H}$), not difficult to see the integration maps $\partial\mathbb{D}$ to the polygon ∂P with the same n angles. Then similar to what we have done in question 1 of tutorial 10 to show it is a conformal mapping from $\mathbb{D}$ to P. Here we compute the length of the edges to show it is a regular polygon. denote the length of the line segment in the image corresponding to the arc γ_k from a_k to a_{k+1} by l_k.

$$\begin{aligned}l_k=|\frac{1}{(-1)^{2/n}}\int_{a_k}^{a_{k+1}}\frac{d\zeta}{(\zeta-a_1)^{2/n}...(\zeta-a_n)^{2/n}}|&=|\frac{1}{(-1)^{2/n}}\int_{\frac{2k\pi}{n}}^{\frac{(2k+2)\pi}{n}}\frac{ie^{i\theta}d\theta}{(e^{i\theta}-a_1)^{2/n}...(e^{i\theta}-a_n)^{2/n}}|\\&=|\int_{\frac{2k\pi}{n}}^{\frac{(2k+2)\pi}{n}}\frac{e^{i\theta}d\theta}{(e^{i\theta}-a_1)^{2/n}...(e^{i\theta}-a_n)^{2/n}}|\\&=|\int_{\frac{2k\pi}{n}}^{\frac{(2k+2)\pi}{n}}\frac{e^{i\theta}d\theta}{(e^{in\theta}-1)^{2/n}}|\end{aligned}$$

Thus as we can see,

$$l_{k+1}=|\int_{\frac{(2k+2)\pi}{n}}^{\frac{(2k+4)\pi}{n}}\frac{e^{i\theta}d\theta}{(e^{in\theta}-1)^{2/n}}|=|\int_{\frac{2k\pi}{n}}^{\frac{(2k+2)\pi}{n}}\frac{e^{i(\theta+2\pi/n)}d\theta}{(e^{in\theta}e^{i2\pi}-1)^{2/n}}|=l_k$$

it is a regular n-gon. And the perimeter is given by

$$\begin{aligned}|n\int_0^{\frac{2\pi}{n}}\frac{e^{i\theta}d\theta}{(e^{in\theta}-1)^{2/n}}|&=|\int_0^{2\pi}\frac{e^{\frac{i\theta}{n}}d\theta}{(e^{i\theta}-1)^{2/n}}|\\&=|\int_0^{\pi}\frac{e^{\frac{i2\theta}{n}}2d\theta}{(e^{i2\theta}-1)^{2/n}}|\\&=|\int_0^{\pi}\frac{e^{\frac{i2\theta}{n}}2d\theta}{(2\sin t)^{2/n}e^{\frac{2it}{n}}}|\\&=2^{\frac{n-2}{n}}\int_0^{\pi}\frac{d\theta}{(\sin t)^{2/n}}\end{aligned}$$