THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4060 Complex Analysis 2022-23
Homework 4 solutions

1. (Exercise 4 in textbook) Recall that $G(z) = i\frac{1-z}{1+z}$ mapping $\mathbb{D}$ to $\mathbb{H}$ conformally. Then the function $F(z) = (G(z) - i)^2$ is a holomorphic surjection from $\mathbb{D}$ to $\mathbb{C}$. The reason why we need to "move it down" is that we need to ensure the holomorphicity near $G(1) = 0$.

2. (Exercise 9 in textbook) In the interior of $\mathbb{D}$, u is a real part of a holomorphic function, this tells us u is harmonic. We only need to show $u(z) = 0$ for $|z| = 1$. In this case $2Re(u(z)) = u(z) + \overline{u(z)} = \frac{i+z}{i-z} + \overline{\frac{i+z}{i-z}} = \frac{i+z}{i-z} + \frac{-i+\frac{1}{z}}{-i-\frac{1}{z}} = 0$ since $|z| = 1 \Rightarrow \bar{z} = \frac{1}{z}$.

3. (Exercise 12 in textbook)

   (a) Suppose the fixed points of f is α and β then we consider the linear fractional transformation $\psi_\alpha(z) = \frac{\alpha - z}{1-\bar{\alpha}z}$, then $\psi_\alpha \circ f \circ \psi_\alpha^{-1}(z)$ maps 0 to 0 and maps $\psi_\alpha(\beta)$ to $\psi_\alpha(\beta)$. Then by Schwarz lemma, we have $\psi_\alpha \circ f \circ \psi_\alpha^{-1}(z) = z \Rightarrow f(z) = z$.

   (b) In the upper half plane, $f(z) = z + a \ \forall a \in \mathbb{R}$ has no fixed point, then the corresponding automorphism of $\mathbb{D}$ also has no fixed point.

   Remark: If we consider the continuous extension to its boundary (including ∞ for $\mathbb{H}$), then there exists at least 1 fixed point since we have the Brouwer fixed-point theorem.

4. (Exercise 13 in textbook)

   (a) We follow the hint, consider the function $\psi_{f(w)} \circ f \circ \psi_w^{-1}$ mapping 0 to 0. Thus by Schwarz lemma, we have:

$$|\psi_{f(w)} \circ f| \leq |\psi_w| \Rightarrow |\frac{f(z) - f(w)}{1 - \overline{f(w)}f(z)}| \leq |\frac{z - w}{1 - \overline{w}z}|$$

   (b) By part (a), we have

$$|\frac{f(z) - f(w)}{z - w}| \cdot |\frac{1}{1 - \overline{f(w)}f(z)}| \leq |\frac{1}{1 - \overline{w}z}|$$

   Let $w \to z$, we get:

$$|f'(z)| \cdot |\frac{1}{1 - |f(z)|^2}| \leq |\frac{1}{1 - |z|^2}|$$