Then clearly, $\boxed{L \geq B \geq \frac{1}{72}}$

Prop 1.10 If $\Omega \supset \mathbb{D}$, f holo on Ω, $f(0)=0$ & $f'(0)=1$, then $\exists\, w_0 \in \mathbb{C}$ s.t. $D(w_0, L) \subset f(\mathbb{D})$.

Pf: Let $\lambda = \lambda(f)$.

Then $\forall n$, $\exists\, w_n \in \mathbb{C}$ s.t. $D(w_n, \lambda - \frac{1}{n}) \subset f(\mathbb{D}) \subset f(\overline{\mathbb{D}})$

By compactness of $f(\overline{\mathbb{D}})$, we may assume $w_n \to w_0 \in f(\overline{\mathbb{D}})$

(by taking subseq w_{n_k} & note that $\lambda - \frac{1}{n_k} \geq \lambda - \frac{1}{k}$)

If $w \in D(w_0, \lambda)$, then $|w - w_0| < \lambda$ and $\exists\, n_0 \geq 1$ s.t.

$|w - w_0| < \lambda - \frac{1}{n_0}$.

Hence, using $w_n \to w_0$, $\exists\, n_1 \geq n_0$ s.t.

$$|w_n - w_0| < \left(\lambda - \frac{1}{n_0}\right) - |w - w_0|, \quad \forall\, n \geq n_1 \geq n_0$$

$$\Rightarrow \quad |w - w_n| \leq |w - w_0| + |w_n - w_0| < \lambda - \frac{1}{n_0} \leq \lambda - \frac{1}{n}$$

$$\Rightarrow \quad w \in D(w_n, \lambda - \tfrac{1}{n}) \subset f(\overline{\mathbb{D}}).$$

Since $w \in D(w_0, \lambda)$ is arbitrary, $D(w_0, \lambda) \subset f(\overline{\mathbb{D}})$.

$$\Rightarrow \quad D(w_0, L) \subset D(w_0, \lambda(f)) \subset f(\overline{\mathbb{D}}). \quad ※$$

Cor 1.11 If f holo. on $\Omega \supset \overline{D(0,R)}$, then $\exists\, w_0 \in \mathbb{C}$ s.t.

$$D(w_0, R|f'(0)|L) \subset f(D(0,R))$$

Pf: Same as the proof of Cor 1.7 ※.

§2. The Little Picard Theorem

Lemma 2.1 Let
- G = simply-connected
- $f : G \to \mathbb{C}$ holo.
- $0, 1 \notin f(G)$

Then $\exists\, g : G \to \mathbb{C}$ holo such that

$$f(z) = -\exp\left(i\pi \cosh(2g(z))\right) \quad \forall z \in G$$

Pf: Since $0 \notin f(G)$ & G simply-connected,

a branch of $\log f(z)$ is well-defined on G.

Let $F(z) = \frac{1}{2\pi i} \log f(z)$.

If $\exists\, z \in G$ s.t. $F(z) = n$ for some $n \in \mathbb{Z}$, then

$$1 = e^{2\pi i n} = e^{\log f(z)} = f(z)$$

which is a contradiction as $1 \notin f(G)$.

$\therefore \quad n \notin F(G), \ \forall n \in \mathbb{Z}$.

In particular, $0, 1 \notin F(G)$.

Hence $\quad H(z) = \sqrt{F(z)} - \sqrt{F(z)-1}$ can be defined

(as G is simply-connected)

Clearly $H(z) \neq 0 \quad \forall z \in G$, and

$$g(z) = \log H(z) \text{ can be defined.}$$

And
$$\cosh(2g) + 1 = \frac{e^{2g} + e^{-2g}}{2} + 1 = \frac{(e^{g} + e^{-g})^2}{2}$$

$$= \frac{(H + \frac{1}{H})^2}{2} = \frac{\left[(\sqrt{F} - \sqrt{F-1}) + (\sqrt{F} + \sqrt{F-1})\right]^2}{2}$$

$$= 2F = \frac{1}{\pi i} \log f$$

$$\Rightarrow \quad f = e^{\pi i} \exp[\pi i \cosh(2g)] = -\exp[\pi i \cosh(2g)] \quad \#$$

Lemma 2.2 Let G, f & g as in Lemma 2.1. Then

$$D(w_0, 1) \setminus g(G) \neq \emptyset \quad \forall w_0 \in \mathbb{C}.$$

Pf: Claim $\forall n \geq 1$, & $m \in \mathbb{Z}$,

$$\pm \log(\sqrt{n} + \sqrt{n-1}) + \tfrac{1}{2} i m \pi \notin g(G).$$

Suppose not, let $g(z) = \pm \log(\sqrt{n}+\sqrt{n-1}) + \frac{1}{2} i m \pi$

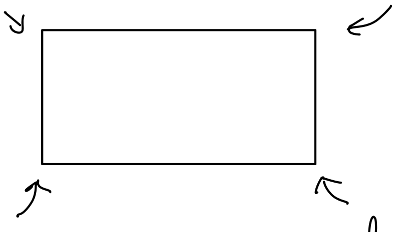
for some $n \geq 1$ & $m \in \mathbb{Z}$.

Then
$$2\cosh(2g(z)) = e^{2g(z)} + e^{-2g(z)}$$
$$= e^{\pm 2\log(\sqrt{n}+\sqrt{n-1})} e^{im\pi} + e^{\mp 2\log(\sqrt{n}+\sqrt{n-1})} e^{-im\pi}$$
$$= e^{im\pi}\left[e^{\pm 2\log(\sqrt{n}+\sqrt{n-1})} + e^{\mp 2\log(\sqrt{n}+\sqrt{n-1})} \right]$$
$$= (-1)^m \left[(\sqrt{n}+\sqrt{n-1})^2 + \frac{1}{(\sqrt{n}+\sqrt{n-1})^2} \right]$$
$$= (-1)^m \left[n + 2\sqrt{n}\sqrt{n-1} + n - 1 + n - 2\sqrt{n}\sqrt{n-1} + n - 1 \right]$$
$$= (-1)^m \, 2(2n-1)$$

$$\Rightarrow \quad f(z) = -e^{i\pi(-1)^m(2n-1)} = -(-1) = 1$$

which is a contradiction as $1 \notin f(G)$

$\log(\sqrt{n}+\sqrt{n-1}) + \frac{1}{2} i (m+1)\pi$ $\quad$ $\log(\sqrt{n+1}+\sqrt{n}) + \frac{1}{2} i (m+1)\pi$



$\log(\sqrt{n}+\sqrt{n-1}) + \frac{1}{2} i m\pi$ $\quad$ $\log(\sqrt{n+1}+\sqrt{n}) + \frac{1}{2} i m\pi$

(reverse horizontally if "−" instead of "+")

height of rectangle $= \frac{\pi}{2} < \sqrt{3}$

width of rectangle $= \log(\sqrt{n+1}+\sqrt{n}) - \log(\sqrt{n}+\sqrt{n-1})$

Note that $\frac{d}{dx}\left[\log(\sqrt{x+1}+\sqrt{x})-\log(\sqrt{x}+\sqrt{x-1})\right]<0$

width of rectangle $\leq \log(\sqrt{1+1}+\sqrt{1})-\log(\sqrt{1}+\sqrt{1-1})$

$$= \log(\sqrt{2}+1) < 1.$$

Hence diagonal of the rectangle $< \left[(\sqrt{3})^2+1\right]^{1/2} = 2$

Therefore, for any $w_0 \in \mathbb{C}$, $D(w_0, 1)$ must contains a point in $\{\pm\log(\sqrt{n}+\sqrt{n-1})+\frac{1}{2}im\pi : n\geq 1, m\in\mathbb{Z}\}$

and hence $D(w_0,1)\setminus g(G)\neq\phi$ #

<u>Thm 2.3</u> <u>Little Picard Thm</u>

If f is an entire function that omits two values, then f is a constant.

<u>Pf</u>: If $f(z)\neq a$ & $f(z)\neq b$, then $h(z)=\frac{f(z)-a}{b-a}\neq 0, 1.$

So we only need consider the case that $\{a,b\}=\{0,1\}$.

Since $\mathbb{C}$ is simply-connected, Lemma 2.2 $\Rightarrow$

$\forall w_0\in\mathbb{C}$, $D(w_0,1)\setminus g(\mathbb{C})\neq\phi$. (as in Lemma 2.2)

Suppose on the contrary that $g\neq$ constant.

Then $g'(z_0) \neq 0$ for some $z_0 \in \mathbb{C}$.

We may assume $z_0 = 0$.

Otherwise, consider $g(z+z_0)$ instead.

Cor 1.11 $\Rightarrow$ $\exists\, w_0 \in \mathbb{C}$ such that

$$D(w_0, 1) \subset g\left(D\left(0, \frac{1}{|g'(0)|L}\right)\right) \subset g(\mathbb{C})$$

which is a contradict.

$\therefore$ $g \equiv$ constant. ※

Great Picard Thm Suppose $z_0 \in \Omega$, f holo on $\Omega \setminus \{z_0\}$ and z_0 is an essential singularity of f. Then in each neighborhood of z_0, f assumes each complex number, with one possible exception, an infinite number of times

Pf: Omitted

Review

Ch1 Preliminaries to Cpx Analysis

Ch2 Cauchy's Thm & Its applications (§5.5 omitted)

- Holomorphic functions defined in term of integrals

$$\int_a^b F(z,s)\,ds$$

- Schwarz reflection principle

Ch3 Meromorphic Functions & the Logarithm

Ch4 Fourier Transform

- Class $\mathfrak{F} = \bigcup_{a>0} \mathfrak{F}_a$
- Estimate of $\hat{f}$ for $f \in \mathfrak{F}$
- Fourier Inversion Formula (for $f \in \mathfrak{F}$)
- Poisson Summation Formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) \quad (\text{for } f \in \mathfrak{F})$$

- Theta function
- Phragmén-Lindelöf Thm (max. principle for unbdd domain)
  (other parts of §3 omitted)

Ch5 Entire Function

- Jensen's Formula
- Functions of Finite Order

$$\rho_f = \inf\{\rho : |f(z)| \le Ae^{B|z|^\rho}, \text{ for some } A \& B\}$$

- Weierstrass Infinite Products &
- Hadamard's Factorization Theorem (for f with $\rho_f < +\infty$)

Ch6 Gamma & Zeta Functions $\Gamma(s)$ & $\zeta(s)$

- Analytic continuations of Gamma & Zeta Functions
- Various properties, formulae, and estimates for $\Gamma(s)$ & $\zeta(s)$

Ch7 Zeta Functions and Prime Number Theorem

- $\pi(x) \sim x/\log x$ as $x \to \infty$

Ch8 Conformal Mappings

- Conformal maps & conformal equivalence
- Angle preserving property
- Explicit conformal map between $\mathbb{D}$ and $\mathbb{H}$
- Fractional linear transformations $z \mapsto \frac{az+b}{cz+d}$
  (translations, rotations, scalings, and inversion),
  maps "lines & circles" to "lines & circles"
- Elementary examples of conformal maps between specific domains.
- Dirichlet problem
- Schwarz Lemma
- Automorphism groups
  $Aut(\mathbb{D})$, $Aut(\mathbb{H})$ (and $Aut_0(\mathbb{D})$)

- Riemann Mapping Theorem
- Normal Family and Montel's Theorem
- Hurwitz Thm (and corresponding Prop 3.5)
- Conformal Maps onto Polygons,
- Continuous extension to the boundary
- Schwarz-Christoffel Integral, Elliptic Integral

Final exam: $\boxed{\text{May 10 (Wednesday) 9:30–11:30 am, multi-purpose hall, CC.}}$

covers all material including those in lectures, tutorials, homework, & textbook (including all exercises in Textbook no matter it's assigned in homework or not) up to ch 8, except Ch 7, with emphasis on those material after the mid-term (i.e. Ch 8). But those material before mid-term (i.e. Ch 1–6) may also be tested directly/explicitly or indirectly/implicitly.