Thm 2.1 (i) If $k \geq 3$, $E_k(\tau)$ converges & is holo. in $\mathbb{H}$.

(ii) If $k = \text{odd}$, $E_k(\tau) = 0$.

(iii) $E_k(\tau+1) = E_k(\tau)$ & $E_k(\tau) = \frac{1}{\tau^k} E_k(\frac{1}{\tau})$ (modular character)

Pf: (i) If $k \geq 3$, Lemma 1.5 & its proof

$\Rightarrow$ $E_k(\tau)$ converges abolutely and uniformly on compact subsets of $\mathbb{H}$.

$\therefore$ $E_k(\tau)$ converges to a holo. on $\mathbb{H}$

(ii) Since $\omega \in \Lambda_\tau^* \Leftrightarrow -\omega \in \Lambda_\tau^*$,

$$E_k(\tau) = \sum_{\omega \in \Lambda_\tau^*} \frac{1}{\omega^k} = \sum_{-\omega \in \Lambda_\tau^*} \frac{1}{(-\omega)^k} = (-1)^k \sum_{-\omega \in \Lambda_\tau^*} \frac{1}{\omega^k} = (-1)^k E_k(\tau)$$

$\therefore$ $k = \text{odd} \Rightarrow E_k(\tau) = 0$

(iii) Using $\Lambda_{\tau+1}^* = \Lambda_\tau^*$, we have

$$E_k(\tau+1) = \sum_{\omega \in \Lambda_{\tau+1}^*} \frac{1}{\omega^k} = \sum_{\omega \in \Lambda_\tau^*} \frac{1}{\omega^k} = E_k(\tau)$$

Using $\omega \in \Lambda_{-\frac{1}{\tau}}^* \Leftrightarrow \tau\omega \in \Lambda_\tau^*$, we have

$$E_k(-\tfrac{1}{\tau}) = \sum_{\omega \in \Lambda_{-\frac{1}{\tau}}^*} \frac{1}{\omega^k} = \sum_{\tau\omega \in \Lambda_\tau^*} \frac{\tau^k}{(\tau\omega)^k} = \tau^k \sum_{\omega' \in \Lambda_\tau^*} \frac{1}{(\omega')^k} = \tau^k E_k(\tau)$$

※

Thm 2.2 $\forall \tau \in \mathbb{H}$,

$$\wp_\tau(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1)\, E_{2k+2}(\tau)\, z^{2k} \qquad \text{near } z=0$$

Pf: For simplicity, τ will be omitted in the following calculation.

By definition,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right]$$

$$= \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right] \qquad (\text{since } \omega \in \Lambda^* \Leftrightarrow -\omega \in \Lambda^*)$$

$$= \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \frac{1}{\omega^2} \left[\frac{1}{\left(1-\frac{z}{\omega}\right)^2} - 1 \right]$$

near $z=0$ $$= \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \frac{1}{\omega^2} \left[\sum_{l=0}^{\infty} (l+1) \left(\frac{z}{\omega}\right)^l - 1 \right]$$

$\therefore$ near $z=0$ $\qquad \left(\frac{1}{1-\xi} = \sum_{l=0}^{\infty} \xi^l \Rightarrow \frac{1}{(1-\xi)^2} = \sum_{l=0}^{\infty} l \xi^{l-1} \right)$

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \sum_{l=1}^{\infty} (l+1) \frac{z^l}{\omega^{l+2}}$$

$$= \frac{1}{z^2} + \sum_{l=1}^{\infty} (l+1) \left(\sum_{\omega \in \Lambda^*} \frac{1}{\omega^{l+2}} \right) z^l \qquad \text{by absolute convergence}$$

$$= \frac{1}{z^2} + \sum_{l=1}^{\infty} (l+1)\, E_{l+2}\, z^l$$

By Thm 2.1, $E_{\text{odd}} = 0$, we have

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1)\, E_{2k+2}\, z^{2k}, \quad \text{near } z=0 .$$

$\#$

Cor 2.3 If $g_2 = 60E_4$ and $g_3 = 140E_6$, then
$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

Pf: By Thm 2.2, near z,

$$\wp(z) = \frac{1}{z^2} + 3E_4 z^2 + 5E_6 z^4 + \cdots$$

$$\Rightarrow (\wp(z))^3 = \left(\frac{1}{z^2} + 3E_4 z^2 + 5E_6 z^4 + \cdots\right)^3$$
$$= \frac{1}{z^6}\left[1 + z^4(3E_4 + 5E_6 z^2 + \cdots)\right]^3$$
$$= \frac{1}{z^6}\left[1 + 3z^4(3E_4 + 5E_6 z^2 + \cdots) + O(|z|^8)\right]$$
$$= \frac{1}{z^6} + \frac{9E_4}{z^2} + 15E_6 + \cdots$$

Also, Thm 2.2 $\Rightarrow$

$$\wp'(z) = \frac{-2}{z^3} + 6E_4 z + 20E_6 z^3 + \cdots$$

$$\Rightarrow (\wp'(z))^2 = \frac{4}{z^6} - \frac{24E_4}{z^2} - 80E_6 + \cdots$$

$$\therefore \quad (\wp')^2 - 4(\wp)^3 = \frac{4}{z^6} - \frac{24E_4}{z^2} - 80E_6 + \cdots$$
$$- \frac{4}{z^6} - \frac{36E_4}{z^2} - 60E_6 - \cdots$$

$$= -60E_4\frac{1}{z^2} - 140E_6 + O(|z|^2)$$

$$(\wp')^2 - 4(\wp)^3 + 60E_4\wp + 140E_6 = 60E_4\left(\wp - \frac{1}{z^2}\right) + O(|z|^2)$$

$$= O(|z|^2)$$

$\therefore$ $(\wp')^2 - 4(\wp)^3 + g_2\wp + g_3$ is holo. near $z=0$ and equals 0 at $z=0$

Since it is also doubly periodic, and holo $\forall z \notin \Lambda$, we have $(\wp')^2 - 4(\wp)^3 + g_2\wp + g_3$ is doubly periodic & holo, $\forall z \in \mathbb{C}$.

By Thm 1.2, it is a constant.

As it equals to 0 at $z=0$, it is constantly zero.

#

<u>Remark</u>: Together with Thm 1.7,

$$4(\wp - e_1)(\wp - e_2)(\wp - e_3) = 4\wp^3 - g_2\wp - g_3$$

$$\Rightarrow \begin{cases} e_1 + e_2 + e_3 = 0, \\ 4(e_1e_2 + e_2e_3 + e_3e_1) = -g_2, \\ 4e_1e_2e_3 = g_3 \end{cases}$$

(And $4e_i^3 - g_2e_i - g_3 = 0$, $i = 1,2,3$)

2.2 Eisenstein Series and Divisor Functions

Lemma 2.4 If $k \geq 2$ & $\operatorname{Im}(\tau) > 0$, then
$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\tau)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{\ell=1}^{\infty} \ell^{k-1} e^{2\pi i \tau \ell}$$

Pf In Ch5, §3.2, it is proved that
$$\pi \cot \pi z = \lim_{N \to +\infty} \sum_{|m| \leq N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} \quad \forall z \notin \mathbb{Z}$$

Since $\operatorname{Im}(\tau) > 0$, we have
$$\pi \cot \pi \tau = \lim_{N \to +\infty} \sum_{|m| \leq N} \frac{1}{\tau+n} = \frac{1}{\tau} + \sum_{n=1}^{\infty} \frac{2\tau}{\tau^2-n^2}$$

Differentiating term-by-term,
$$-\frac{\pi^2}{\sin^2 \pi\tau} = -\frac{1}{\tau^2} - \sum_{n=1}^{\infty} \frac{2(\tau^2+n^2)}{(\tau^2-n^2)^2}$$
$$= -\frac{1}{\tau^2} - \sum_{n=1}^{\infty} \left[\frac{1}{(\tau-n)^2} + \frac{1}{(\tau+n)^2}\right]$$
$$= -\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^2}$$

$$\therefore \quad \sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^2} = \frac{\pi^2}{\sin^2 \pi\tau} = \frac{(2\pi i)^2}{(e^{i\pi\tau} - e^{-i\pi\tau})^2}$$
$$= \frac{(-2\pi i)^2}{e^{-2\pi i\tau}(1-e^{2\pi i\tau})^2} = (-2\pi i)^2 \frac{e^{2\pi i\tau}}{(1-e^{2\pi i\tau})^2}$$

$$= (-2\pi i)^2 \sum_{l=1}^{\infty} l e^{2\pi i \tau l}$$

(we've used formula $\frac{\omega}{(1-\omega)^2} = \sum_{l=1}^{\infty} l\omega^l$, $|\omega| = |e^{2\pi i \tau}| = e^{-2\pi \operatorname{Im}\tau} < 1$)

$\therefore$ The formula holds for $k=2$.

Then induction $\Rightarrow$ The formula holds $\forall\, k \geq 2$ (diff. term-by-term).

$\divideontimes$

Def: Divisor Functions

$$\sigma_l(r) = \sum_{d|r} d^l \qquad \text{for } r, l \in \mathbb{N}$$

Thm 2.5 If $k \geq 4$, k even, and $\tau \in \mathbb{H}$, then

$$E_k(\tau) = 2\zeta(k) + \frac{2(-1)^{\frac{k}{2}}(2\pi)^k}{(k-1)!} \sum_{r=1}^{\infty} \sigma_{k-1}(r) e^{2\pi i \tau r}$$

Pf: Note:
- $\operatorname{Im}(\tau) \geq t_0 > 0 \Rightarrow |e^{2\pi i \tau r}| \leq e^{-2\pi t_0 r}$
- $\sigma_{k-1}(r) = \sum_{d|r} d^{k-1} \leq r \cdot r^{k-1} = r^k$.

less than r factors

$$\therefore \sum_{r=1}^{\infty} |\sigma_{k-1}(r) e^{2\pi i \tau r}| \leq \sum_{r=1}^{\infty} r^k e^{-2\pi t_0 r}$$

convergent because of the exponential decay.

$\Rightarrow \sum_{r=1}^{\infty} \sigma_{k-1}(r) e^{2\pi i \tau r}$ is absolutely convergent in $\{\operatorname{Im}\tau \geq t_0\}$, $\forall\, t_0 > 0$.

Now $\displaystyle E_k(\tau) = \sum_{(n,m)\neq(0,0)} \frac{1}{(n+m\tau)^k}$

$$= \sum_{n\neq 0} \frac{1}{n^k} + \sum_{m\neq 0} \sum_{n=-\infty}^{\infty} \frac{1}{(n+m\tau)^k}$$

$$= 2\zeta(k) + \sum_{m>0} \sum_{n=-\infty}^{\infty} \frac{1}{(n+m\tau)^k} + \sum_{m>0} \sum_{n=-\infty}^{\infty} \frac{1}{(n-m\tau)^k}$$

$$= 2\zeta(k) + \sum_{m>0} \sum_{n=-\infty}^{\infty} \frac{1}{(n+m\tau)^k} + \sum_{m>0} \sum_{n=-\infty}^{\infty} \frac{1}{(-n+m\tau)^k}$$

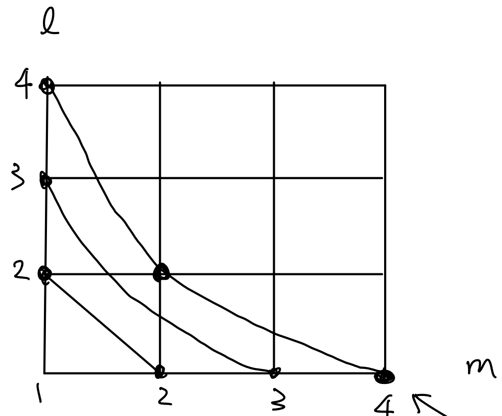
(k even)

$$= 2\zeta(k) + 2\sum_{m>0} \sum_{n=-\infty}^{\infty} \frac{1}{(n+m\tau)^k}$$

$$= 2\zeta(k) + 2\sum_{m>0} \frac{(-2\pi i)^k}{(k-1)!} \sum_{l=1}^{\infty} l^{k-1} e^{2\pi i(\tau m)l} \qquad \text{(Lemma 2.4)}$$

$$= 2\zeta(k) + 2\,\frac{(-1)^{\frac{k}{2}}(2\pi)^k}{(k-1)!} \sum_{m>0} \sum_{l=1} l^{k-1} e^{2\pi i\tau(ml)}$$

$\displaystyle\sum_{m>0}\sum_{l=1}$ are sum for :



$r = 4 = 4\times1,\ 2\times2,\ 1\times4$

Let $r = ml$

Then the sum is $\displaystyle\sum_{r=1}^{\infty}\left(\sum_{l\mid r} l^{k-1}\right) e^{2\pi i\tau r}$

$\therefore \quad E_k(\tau) = 2\zeta(k) + 2\,\dfrac{(-1)^{\frac{k}{2}}(2\pi)^k}{(k-1)!}\displaystyle\sum_{r=1}^{\infty}\sigma_{k-1}(r)\,e^{2\pi i \tau r}$ ※

<u>Remark</u>: Note that E_2 is not defined, as $\displaystyle\sum_{(n,m)\neq(0,0)}\frac{1}{(n+m\tau)^2}$ not converges <u>absolutely</u>. However, we can consider

$$F(\tau) = \sum_{\substack{m \\ (n,m)\neq(0,0)}}\left(\sum_n \frac{1}{(n+m\tau)^2}\right).$$

(<u>forbidden Eisenstein series</u>)

Then
$$F(\tau) = \sum_{n\neq 0}\frac{1}{n^2} + \underbrace{\sum_{m\neq 0}\sum_{n=-\infty}^{\infty}\frac{1}{(n+m\tau)^2}}_{\text{convergent}}$$

Lemma 2.4

$$= 2\zeta(2) + 2\sum_{m>0}\left[(-2\pi i)^2\sum_{l=1}^{\infty} l\, e^{2\pi i \tau m l}\right]$$

$$= 2\zeta(2) - 2(2\pi)^2\sum_{r=1}^{\infty}\sigma_1(r)\,e^{2\pi i \tau r}$$

same formula works and we have

<u>Cor 2.6</u> $F(\tau) = \displaystyle\sum_{\substack{m \\ (n,m)\neq(0,0)}}\left(\sum_n \frac{1}{(n+m\tau)^2}\right)$ converges and

$$F(\tau) = 2\zeta(2) - 8\pi^2\sum_{r=1}^{\infty}\sigma_1(r)\,e^{2\pi i \tau r}$$

<u>Remark</u>: As the convergent is not absolute, F may not have all the properties of E_k. Eg: $F(-\frac{1}{\tau})\tau^{-2} \neq F(\tau)$.

Addition Topic : Range of an Analytic Function

(Ref: John Conway, Functions of one complex variables)

§1 Bloch's Theorem

<u>Lemma 1.1</u> Suppose $\begin{cases} \bullet\ f \text{ holo. on } \mathbb{D} \\ \bullet\ f(0)=0 \\ \bullet\ f'(0)=1 \\ \bullet\ |f(z)| \leq M \quad \forall z \in \mathbb{D} \end{cases}$

Then $\begin{cases} \bullet\ M \geq 1 \quad \text{and} \\ \bullet\ D(0, \frac{1}{6M}) \subset f(\mathbb{D}) \end{cases}$

<u>Pf:</u> $f(0)=0$ & $f'(0)=1$

$$\Rightarrow \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \text{where } a_n = \frac{f^{(n)}(0)}{n!} .$$

Using $|f| \leq M$, and Cauchy integral famula, we have

$$\forall\, 0<r<1, \quad |a_n| = \frac{|f^{(n)}(0)|}{n!} \leq \frac{\sup|f|}{r^n} \leq \frac{M}{r^n} \qquad \forall\, n \geq 2$$

letting $r \to 1$, we have $|a_n| \leq M$, $\forall\, n \geq 2$.

Clearly, same argument gives

$$1 \leq M \qquad (\text{as } a_n = 1 \text{ for } n=1)$$

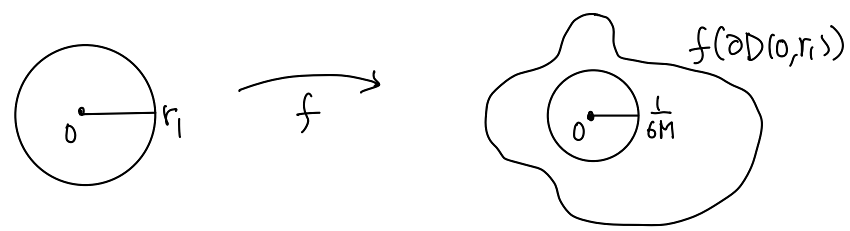
Let $r_1 = \frac{1}{4M} < \frac{1}{4}$. Then for z with $|z| = r_1$

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} |a_n||z|^n \geq |z| - M \sum_{n=2}^{\infty} |z|^n$$

$$= r_1 - M \sum_{n=2}^{\infty} r_1^n = r_1 - \frac{M r_1^2}{1 - r_1}$$

$$= \frac{1}{4M} - \frac{1}{16M - 4} \geq \frac{1}{4M} - \frac{1}{12M} \quad (\text{since } M \geq 1)$$

$$= \frac{1}{6M}$$



$\forall\, w \in D(0, \frac{1}{6M})$ (regarded as points in the target plane),

consider $g(z) = f(z) - w$ on $D(0, r_1)$.

By the above estimate, for $z \in \partial D(0, r_1)$

$$|f(z) - g(z)| = |w| < \frac{1}{6M} \leq |f(z)|.$$

Hence Rouché's Thm $\Rightarrow$

f & g have same number of zeros in $D(0, r_1)$.

$\Rightarrow \exists$ at least one $z_0 \in D(0, r_1)$ s.t. $g(z_0) = 0$ since $f(0) = 0$.

$\therefore\ w \in f(D(0, r_1)) \subset f(\mathbb{D})$.

Since $w \in D(0, \frac{1}{6M})$ is arbitrary, $D(0, \frac{1}{6M}) \subset f(\mathbb{D})$. #

Lemma 1.2 If
$$\begin{cases} \bullet\ g: D(0,R) \to \mathbb{C} \text{ holo} \\ \bullet\ g(0)=0 \\ \bullet\ |g'(0)| = \mu > 0 \\ \bullet\ |g(z)| \leq M \quad \forall z \in D(0,R) \end{cases}$$
then
$$D\left(0, \frac{R^2\mu^2}{6M}\right) \subset g(D(0,R))$$

Pf: Let $f(z) = \dfrac{g(Rz)}{Rg'(0)} : \mathbb{D} \to \mathbb{C}$.

Then $f(0) = 0$, $f'(0) = 1$ & $|f(z)| \leq \dfrac{M}{R\mu}$.

Lemma 1.1 $\Rightarrow$ $D\left(0, \dfrac{R\mu}{6M}\right) \subset f(\mathbb{D})$

Note that $\forall\, w \in D\left(0, \dfrac{R^2\mu^2}{6M}\right)$, $\dfrac{w}{Rg'(0)} \in D\left(0, \dfrac{R\mu}{6M}\right)$

$\Rightarrow$ $\dfrac{w}{Rg'(0)} \in f(\mathbb{D})$

$\Rightarrow$ $w = Rg'(0)f(\zeta)$ for some $\zeta \in \mathbb{D}$

$= g(R\zeta)$.

Note that $R\zeta \in D(0,R)$, we have $w \in g(D(0,R))$

Hence $D\left(0, \dfrac{R^2\mu^2}{6M}\right) \subset g(D(0,R))$ #

Lemma 1.3 If
- $f : D(a,r) \to \mathbb{C}$ holo. such that
- $|f'(z) - f'(a)| < |f'(a)| \quad \forall z \in D(a,r) \setminus \{a\}$

Then f is 1-1

Pf: For $z_1 \neq z_2 \in D(a,r)$, let γ = line segment joining z_1, z_2.

Then $\gamma \subset D(a,r)$ &

$$|f(z_1) - f(z_2)| = \left| \int_\gamma f'(z)\,dz \right|$$

$$\geq \left| \int_\gamma f'(a)\,dz \right| - \left| \int_\gamma [f'(z) - f'(a)]\,dz \right|$$

$$\geq |f'(a)|\,|z_1 - z_2| - \int_\gamma |f'(z) - f'(a)|\,|dz|$$

$$> 0 \quad \text{by assumption}$$

※

Thm 1.4 (Block Thm) Let
$$\begin{cases} \bullet\ f : \Omega \to \mathbb{C} \text{ holo,} \\ \bullet\ \overline{\mathbb{D}} \subset \Omega \\ \bullet\ f(0) = 0 \ \ \& \ \ f'(0) = 1 \end{cases}$$

Then $\exists$ a disk $D \subset \mathbb{D}$ s.t.

$$\begin{cases} \bullet\ f|_D \text{ is 1-1 and} \\ \bullet\ f(D) \supset D(w_0, \frac{1}{72}) \text{ for some } w_0 \in \mathbb{C} \end{cases}$$

Pf: Let $K(r) = \max\{ |f'(z)| : |z| = r \}$, &

$$h(r) = (1-r)K(r)$$

Then $h: [0,1] \to \mathbb{R}$ is continuous

$$h(0) = 1, \quad h(1) = 0.$$

Let $r_0 = \sup\{ r : h(r) = 1 \}$,

then $h(r_0) = 1$ and $h(r) < 1, \forall r > r_0$.

On $\{|z| = r_0\}$ choose a max. point a such that

$$|f'(a)| = K(r_0) \qquad (|a| = r_0).$$

Then $|f'(a)| = \dfrac{1}{1-r_0}$ (using $h(r_0) = 1$)

Consider the disk $D(a, \rho_0)$ where $\rho_0 = \dfrac{1-r_0}{2}$

Then $\forall z \in D(a, \rho_0)$,

$$|z| \leq |a| + |z-a| < r_0 + \rho_0 = \frac{1+r_0}{2}$$

By max. module principle,

$$|f'(z)| \leq K\left(\frac{1+r_0}{2}\right) = \frac{1}{1 - \frac{1+r_0}{2}} h\left(\frac{1+r_0}{2}\right) < \frac{2}{1-r_0} = \frac{1}{\rho_0},$$

since $h\left(\frac{1+r_0}{2}\right) < 1$ by the construction of r_0 & $r_0 < \frac{1+r_0}{2}$.

Therefore $|f'(z) - f'(a)| \leq |f'(z)| + |f'(a)| < \dfrac{1}{\rho_0} + \dfrac{1}{2\rho_0} = \dfrac{3}{2\rho_0}$.

Applying Schwarz's Lemma to

$$F(\zeta) = \frac{2\rho_0}{3}\left[f'(\rho_0\zeta + a) - f'(a)\right] \quad \text{for } \zeta \in \mathbb{D},$$

we have $\quad |f'(z) - f'(a)| \le \frac{3}{2\rho_0^2}|z-a| \quad , \forall z \in D(a, \rho_0)$

If $z \in D = D(a, \frac{\rho_0}{3})$, then

$$|f'(z) - f'(a)| < \frac{3}{2\rho_0^2} \cdot \frac{\rho_0}{3} = \frac{1}{2\rho_0} = |f'(a)|$$

By Lemma 1.4. $f|_D$ is 1-1.

For the 2nd statement, let

$$g(z) = f(z+a) - f(a) \quad \text{on } D(0, \frac{\rho_0}{3})$$

Then $g(0) = 0$, $\quad |g'(0)| = |f'(a)| = \frac{1}{2\rho_0}$

Moreover, $\quad |g(z)| = |f(z+a) - f(a)| \le \int_\gamma |f'(w)||dw|$

where γ = line segment joining a & $z+a$ in $D = D(a, \frac{\rho_0}{3}) \subset D(a, \rho_0)$

$$\therefore \ |f'(w)| \le \frac{1}{\rho_0}, \ \forall w \text{ on } \gamma.$$

Hence $\quad |g(z)| \le \frac{1}{\rho_0} \text{ length}(\gamma) = \frac{1}{\rho_0}|z| < \frac{1}{3}$

Applying Lemma 1.2 to g, we have

$$g(D(0, \tfrac{\rho_0}{3})) \supset D(0, \frac{(\frac{\rho_0}{3})^2 (\frac{1}{2\rho_0})^2}{6 \cdot \frac{1}{3}}) = D(0, \tfrac{1}{72}).$$

$$\Rightarrow \quad f(D) = f(D(a, \tfrac{\rho_0}{3})) \supset D(f(a), \tfrac{1}{72}) = D(w_0, \tfrac{1}{72})$$

where $w_0 = f(a)$ #

Cor 1.7 If $\begin{cases} \bullet\ \Omega \supset \overline{D(0,R)}, \\ \bullet\ f \text{ holo on } \Omega \end{cases}$

Then $f(\overline{D(0,R)}) \supset D(w_0, \frac{R|f'(0)|}{72})$ for some $w_0 \in \mathbb{C}$.

Pf: Applying Bloch's Thm to $\frac{f(Rz)-f(0)}{Rf'(0)}$ on $\overline{\mathbb{D}}$ if $f'(0) \neq 0$.

If $f'(0)=0$, then the result is trivial as $f(\overline{D(0,R)}) \supset \{f(0)\}$. ※

Def 1.8 Bloch's constant is defined by

$$B = \inf \{ \beta(f) : f \in \mathcal{F} \}$$

where $\begin{cases} \mathcal{F} = \{ f : f \text{ holo. on some } \Omega \supset \overline{\mathbb{D}} \text{ s.t. } f(0)=0 \ \& \ f'(0)=1 \} \\ \beta(f) = \sup \left\{ r : \begin{array}{l} \exists\, D \subset \overline{\mathbb{D}} \text{ and } w_0 \in \mathbb{C} \text{ s.t.} \\ f|_D \text{ is 1-1 and } D(w_0, r) \subset f(D) \end{array} \right\} \end{cases}$

Then Bloch's Thm can be rephased as $\boxed{B \geq \frac{1}{72}}$

Def 1.9: Landau's constant $L = \inf\{ \lambda(f) : f \in \mathcal{F} \}$

where $\mathcal{F}$ as in Def 1.8 and

$$\lambda(f) = \sup\{ r : \exists\, w_0 \in \mathbb{C} \text{ s.t. } D(w_0, r) \subset f(\mathbb{D}) \}$$