Lemma 1.9 Every even elliptic function F with periods 1 & τ is a rational function of $\wp$.

Pf: Step 1 May assume F has no zeros or poles on Λ, the lattice generated by 1 & τ.

Pf of Step 1: If F has a zero or pole at some point in Λ, then periodicity $\Rightarrow$ 0 is a zero or pole of F.

F is even $\Rightarrow$ 0 is of even order $= 2k$

Since $\wp$ has a pole of order 2 at 0,

$F\wp^{\pm k}$ ($\pm$ depending on 0 is a zero or a pole)

is an even elliptic function with no zero or pole on Λ.

If $F\wp^{\pm k} = R(\wp)$ is a rational function of $\wp$, then so is $F = \wp^{\mp k} R(\wp)$.

Step 2 Suppose F has no zero or pole on Λ. Then F is a rational function of $\wp$. And the proof of the lemma is completed.

Pf of step 2 : If $a \notin \Lambda$ is a zero of F, then

$-a \notin \Lambda$ & is a zero of F since F is even.

Note that $a \sim -a \Leftrightarrow a-(-a) \in \Lambda$

$\Leftrightarrow 2a \in \Lambda$

$\therefore$ If $a \in P_0$, then $a \sim -a \Leftrightarrow a = \frac{1}{2}, \frac{\tau}{2}, \text{or } \frac{1+\tau}{2}$.

If $a = \frac{1}{2}, \frac{\tau}{2}, \text{or } \frac{1+\tau}{2}$, then

$\wp(z) - \wp(a)$ has a double zero at a (and $-a$)

$\Rightarrow \left(\wp(z)-\wp(a)\right)^{\frac{1}{2}\text{order}_F(a)}$ and F have an zero of the same order at a.

If $a \neq \frac{1}{2}, \frac{\tau}{2}, \& \frac{1+\tau}{2}$, then

$\wp(z) - \wp(a)$ has a simple zero at a (and $-a$)

$\Rightarrow \left(\wp(z)-\wp(a)\right)^{\text{order}_F(a)}$ and F have an zero of the same order at a.

If $a_1, \cdots, a_m$ are zeros of F, then

$$\prod_i \left[\wp(z) - \wp(a_i)\right]^{k_i} \quad \text{with } k_i = \begin{cases} \frac{1}{2}\text{order}_F(a_i), & a_i = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \\ \text{order}_F(a_i), & \text{otherwise} \end{cases}$$

has the same zeros as F counting multiplicity.

Similarly, if $b_1, \dots, b_m$ are poles of F, then

$$\prod_j \left[\wp(z) - \wp(b_j)\right]^{-l_j} \quad \text{with } l_j = \begin{cases} \frac{1}{2}\,\text{order}_F(b_j), & b_j = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \\ \text{order}_F(b_j), & \text{otherwise} \end{cases}$$

has the same poles as F counting multiplicity.

Therefore F and

$$G = \frac{\prod_i \left[\wp(z) - \wp(a_i)\right]^{k_i}}{\prod_j \left[\wp(z) - \wp(b_j)\right]^{l_j}}$$

has same zeros and poles in P_0.

$\Rightarrow \frac{F}{G}$ is holo and has periods 1 & τ.

(since G is clearly has periods 1 & τ)

Hence Thm 1.2 $\Rightarrow F = cG$ for some constant c.

$\therefore F$ is a rational function of $\wp$. #

Thm 1.8 Every elliptic function f with periods 1 & i is a rational function of $\wp$ and $\wp'$.

Pf: $f = f_{even} + f_{odd}$, with

$$\begin{cases} f_{even}(z) = \dfrac{f(z)+f(-z)}{2} & \text{is even elliptic } (\Omega \equiv 0) \\[2ex] f_{odd}(z) = \dfrac{f(z)-f(-z)}{2} & \text{is odd elliptic } (\Omega \equiv 0) \end{cases}$$

Observe also that $\dfrac{f_{odd}}{\wp'}$ is even elliptic

Applying Lemma 1.9 to f_{even} and $\dfrac{f_{odd}}{\wp'}$, it is clear that f is a rational function of $\wp$ & $\wp'$. #

(In fact $f = R_1(\wp) + \wp' R_2(\wp)$, where R_1, R_2 are rational functions)

2. Modular Character of Elliptic Functions & Eisenstein Series

Note: $\wp$ depends on $\tau \in \mathbb{H}$ and will be denoted by $\wp_\tau$ in this section as we will study how $\wp_\tau$ depends on τ. (Modular Character of $\wp$)

Easy Facts:

$$\boxed{\begin{aligned} &(1)\quad \wp_\tau(z) = \wp_{\tau+1}(z) \\ &(2)\quad \wp_{-\frac{1}{\tau}}(z) = \tau^2 \wp_\tau(\tau z) \qquad (\text{Note: } \tau \in \mathbb{H} \Leftrightarrow -\tfrac{1}{\tau} \in \mathbb{H}) \end{aligned}}$$

Pf of (1): $\forall\, n, m \in \mathbb{Z}$, $\quad n + m(\tau+1) = (n+m) + m\tau$

$$\Rightarrow \quad \Lambda_{\tau+1} = \Lambda_\tau$$

Also, $\quad (n+m, m) = (0,0) \Leftrightarrow (n,m) = (0,0)$

$$\therefore \quad \Lambda^*_{\tau+1} = \Lambda^*_\tau$$

Therefore,
$$\wp_{\tau+1}(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*_{\tau+1}} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right]$$
$$= \frac{1}{z^2} + \sum_{\omega \in \Lambda^*_\tau} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right] \qquad \text{(by abs. convergence)}$$
$$= \wp_\tau(z)$$

Pf of (2) $\quad \forall\, n, m \in \mathbb{Z} \qquad n + m\left(-\frac{1}{\tau}\right) = \frac{-m + n\tau}{\tau}$.

$$\therefore \quad \omega \in \Lambda^*_{-\frac{1}{\tau}} \Leftrightarrow \tau\omega \in \Lambda^*_\tau$$

$$\Rightarrow \quad \wp_{-\frac{1}{\tau}}(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*_{\frac{1}{\tau}}} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right]$$

$$= \frac{1}{z^2} + \sum_{\tau\omega \in \Lambda^*_{\tau}} \left[\frac{\tau^2}{(\tau z + \tau\omega)^2} - \frac{\tau^2}{(\tau\omega)^2} \right]$$

$$= \tau^2 \left\{ \frac{1}{(\tau z)^2} + \sum_{\omega' \in \Lambda^*_{\tau}} \left[\frac{1}{(\tau z + \omega')^2} - \frac{1}{\omega'^2} \right] \right\}$$

$$= \tau^2 \wp_{\tau}(\tau z) \quad \#$$

<u>Def</u>: <u>Modular group</u> = group of transfamations of $\mathbb{H}$ generated by $\tau \mapsto \tau + 1$ & $\tau \mapsto -\frac{1}{\tau}$

<u>Notes</u>: Modular group is a subgroup of $\mathrm{Aut}(\mathbb{H})$

as $\tau + 1 = \dfrac{1\cdot\tau + 1}{0\cdot\tau + 1}$ & $-\dfrac{1}{\tau} = \dfrac{0\cdot\tau + (-1)}{1\cdot\tau + 0}$

2.1 Eisenstein Series

<u>Def</u> The <u>Eisenstein series</u> of <u>order k</u> is defined by, $\forall\, \tau \in \mathbb{H}$,

$$E_k(\tau) = \sum_{(n,m) \neq (0,0)} \frac{1}{(n + m\tau)^k} = \sum_{\omega \in \Lambda^*_{\tau}} \frac{1}{\omega^k}$$

where $\Lambda^*_\tau = \Lambda_\tau \setminus \{(0,0)\}$ & Λ_τ = lattice generated by 1 & τ