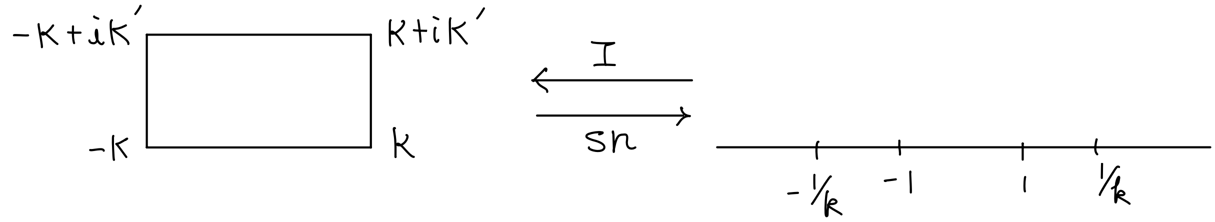
Ch 9 Elliptic Functions

Recall: The elliptic integral

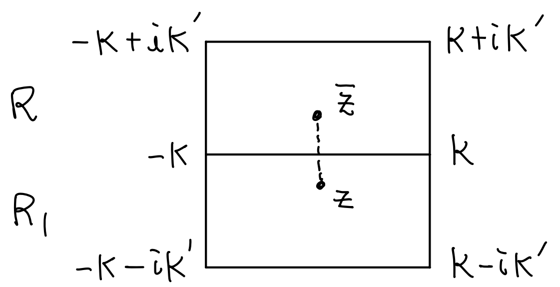
$$I(z) = \int_0^z \frac{d\zeta}{\left[(1-\zeta^2)(1-k^2\zeta^2)\right]^{1/2}} \qquad (0<k<1)$$

gives a map conformal map of $\mathbb{H}$ to the interior of the rectangle R with vertices $-K, K, K+iK', -K+iK'$



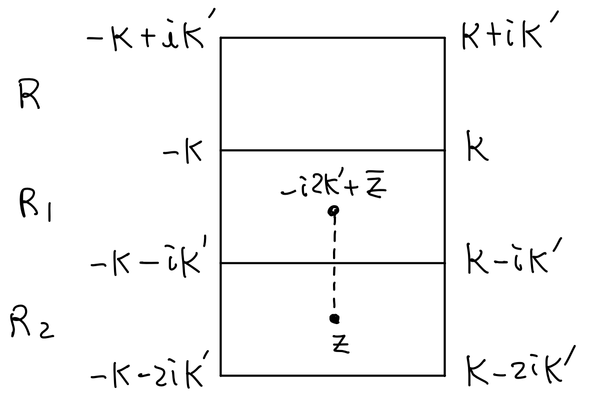
Let $sn = I^{-1} : R \longrightarrow \mathbb{H}$ be the inverse conformal map.

Then boundary straight line segments are map into real axis, Schwarz reflection principle extend $sn(z)$ analytically to



by $sn(z) = \overline{sn(\bar{z})}$ for any z in the reflected rectangle.

Again the extended $sn(z)$ maps boundary straight line segments into real axis, Schwarz reflection principle implies $sn(z)$ is extended analytically to



For $z \in R_2$,

$$\mathrm{sn}(z) = \overline{\mathrm{sn}(-i2K' + \bar{z})}$$

$$= \mathrm{sn}(i2K' + z)$$

(typo in Textbook)

$(z \in R_2 \Rightarrow -i2K' + \bar{z} \in R_1 \Rightarrow i2K' + z \in R)$

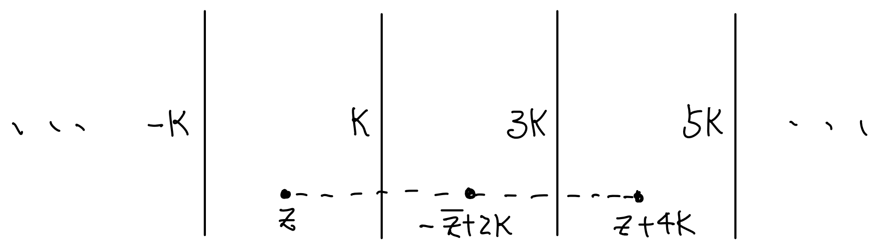
Note that $\mathrm{sn}(z)$ has a pole in the interior of $R \cup R_1 \cup R_2$ as $I(\infty) \in (-K+iK', K+iK')$ and hence a point on $(-K-iK', K-iK')$ maps back to ∞.

And so on, and also reflect upward, $\mathrm{sn}(z)$ is analytically continuated to the infinite strip

$$\mathrm{sn} : \{-K < \mathrm{Re}(z) < K\} \longrightarrow \mathbb{C}$$

such that boundary vertical lines are maps into real-axis.

Applying reflection principle again but horizontally



$$\mathrm{sn}(z) = \overline{\mathrm{sn}(-\bar{z}+2K)} = \mathrm{sn}(z+4K)$$

$sn(z)$ is extended to a meromorphic functions on $\mathbb{C}$ with

$$\begin{cases} sn(z) = sn(z + 2iK') \\ sn(z) = sn(z + 4K) \end{cases}$$

§1 Elliptic Functions

Def: A function f with 2 periods ω_1 and ω_2, i.e.

$$f(z+\omega_1) = f(z) \quad \& \quad f(z+\omega_2) = f(z) \qquad \forall z \in \mathbb{C}$$

is said to be doubly periodic.

Remark: If ω_1 & ω_2 are linear dependent over $\mathbb{R}$ is uninteresting.

In fact, if $\omega_2/\omega_1 \in \mathbb{R}$ (note: $\omega_1, \omega_2 \neq 0$ as periods), then

either f is periodic with a simple period $(\omega_2/\omega_1 \in \mathbb{Q})$

or $f \equiv$ const. $(\omega_2/\omega_1 \in \mathbb{R} \setminus \mathbb{Q})$

Hence, we always assume without loss of generality that

$\operatorname{Im} \tau > 0$, i.e. $\tau \in \mathbb{H}$ (interchanging ω_1 & ω_2, if need.)

Note: f has periods ω_1 & ω_2

$\Rightarrow F(z) \overset{\text{def}}{=\!=} f(\omega_1 z)$ has period 1 & τ

$$\left(\begin{array}{l} F(z+1) = f(\omega_1 z + \omega_1) = f(\omega_1 z) = F(z) \\ F(z+\tau) = f(\omega_1 z + \omega_1 \tau) = f(\omega_1 z + \omega_2) = f(\omega_1 z) = F(z) \end{array}\right)$$

And f meromorphic $\Leftrightarrow$ F meromorphic

$\therefore$

$$\boxed{\text{One only need to study doubly periodic meromorphic functions with periods } 1 \text{ and } \tau \text{ with } \mathrm{Im}(\tau) > 0.}$$

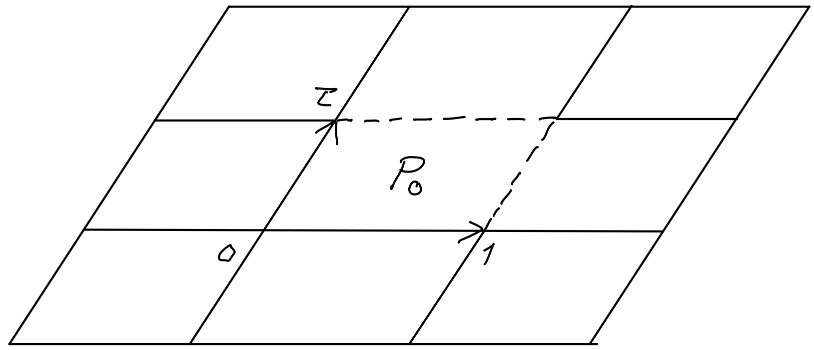
Def: The set $\Lambda = \{n + m\tau : n, m \in \mathbb{Z}\}$ is called the <u>lattice generated</u> by 1 and τ

Def: The <u>fundamental parallelogram</u> associated to the lattice Λ is

$$P_0 = \{z \in \mathbb{C} : z = a + b\tau, \text{ where } 0 \leq a < 1 \text{ \& } 0 \leq b < 1\}$$

A <u>period parallelogram</u> P is any translation of P_0:

$$P = P_0 + h, \text{ with } h \in \mathbb{C}.$$



<u>Easy Facts</u>: (<u>Prop. 1.1</u>)

(1) If f is doubly periodic with periods 1 & τ, then

$$f(z+\lambda) = f(z) \quad \forall \lambda \in \Lambda, \ \forall z \in \mathbb{C}$$

$$(\text{i.e. } f(z+m+n\tau) = f(z), \ \forall m,n \in \mathbb{Z}, \ \forall z \in \mathbb{C})$$

$\therefore$ f is constant under translations by elements in Λ.

i.e. if $z \sim w$ <u>congruent modulo Λ</u>

(where $z \sim w \bmod \Lambda \Leftrightarrow z - w \in \Lambda$)

then $f(z) = f(w)$

(2) Any $z \in \mathbb{C}$, $\exists$ unique pt. $w \in P_0$ s.t.

$$z \sim w \mod \Lambda.$$

Same conclusion for any period parallelogram $P = P_0 + h$.

<u>Pf</u>: Let $z \in \mathbb{C}$.

$\mathrm{Im}(\tau) > 0 \Rightarrow \exists\, a, b \in \mathbb{R}$ s.t. $z = a + b\tau$

$\Rightarrow \quad z = (a - [a]) + (b - [b])\tau + ([a] + [b]\tau)$

where $[a]$ = largest integer $\leq a$,
same for b.

Then $w = (a - [a]) + (b - [b])\tau \in P_0$ s.t.

$z - w = [a] + [b]\tau \in \Lambda$.

For uniqueness, if $w_1, w_2 \in P_0$ s.t. $z \sim w_1$ & $z \sim w_2$.

Then $w_1 \sim w_2$, i.e. $w_1 - w_2 \in \Lambda$.

Note that if $w_i = a_i + b_i\tau$, $i = 1, 2$

then $w_1 - w_2 = (a_1 - a_2) + (b_1 - b_2)\tau \in \Lambda$

Hence $a_1 = a_2 + n$ and $b_1 = b_2 + m$

for some $n, m \in \mathbb{Z}$.

But $w_i \in P_0 \Rightarrow 0 \leq a_i < 1$ & $0 \leq b_i < 1$

$\Rightarrow \quad |a_1 - a_2| < 1$ & $|b_1 - b_2| < 1$

$\Rightarrow \quad n, m = 0$

$\therefore \quad w_1 = w_2$.

Applying above to $z + h$, we have the result for P ※

(3) If f is doubly periodic with periods 1 & τ, then f is uniquely determined by $f|_P$ for any $P = P_0 + h$.

Pf: Easy consequence of (1) & (2).

(4) $$\mathbb{C} = \bigcup_{n,m\in\mathbb{Z}} (n + m\tau + P_0)$$

$\therefore$ $\mathbb{C}$ is covered by period parallelograms (translated by lattice points) and the union is disjoint

1.1 Liouville's Theorem

Thm 1.2 An entire doubly periodic function is constant.

Pf: By Prop 1.1 (Easy facts), $\sup_{\mathbb{C}} |f| = \sup_{P_0} |f|$

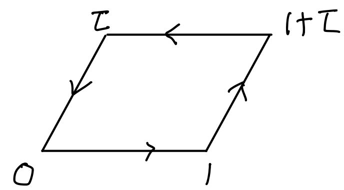
If f is entire, $\sup_{P_0} |f|$ is finite. Hence f is bdd on $\mathbb{C}$. Liouville's Thm $\Rightarrow$ $f \equiv$ const. ※

So we define

Def: A non-constant doubly periodic meromaphic function is called an elliptic function.

Thm 1.3 The total number (counting multiplicity) of poles of an elliptic function in P_0 is always ≥ 2

Pf: Suppose that f has no poles on ∂P_0.



Then

$$2\pi i \sum \operatorname{Res} f = \int_{\partial P_0} f$$

$$= \left(\int_0^1 + \int_1^{1+\tau} + \int_{1+\tau}^{\tau} + \int_{\tau}^{0} \right) f(z)\,dz$$

Note that

$$\int_{1+\tau}^{\tau} f(z)\,dz = \int_1^0 f(z+\tau)\,dz = \int_1^0 f(z)\,dz \qquad (\tau \text{ is a period})$$

$$\int_1^{1+\tau} f(z)\,dz = \int_0^{\tau} f(z+1)\,dz = \int_0^{\tau} f(z)\,dz \qquad (1 \text{ is a period})$$

$\therefore \quad 2\pi i \sum \operatorname{Res} f = 0$

$\Rightarrow$ f has at least 2 poles (counting multiplicity),

Otherwise $\sum \operatorname{Res} f \neq 0$.

If f has a pole on P_0, then choose $h \in \mathbb{C}$ small enough such that f has no pole on ∂P, where $P = P_0 + h$.

Same argument shows that f has at least 2 poles. #

Thm 1.4 Every elliptic function of order m has m zeros in P_0.

(The total number of poles (counting multiplicity) of an elliptic function is called its order.)

Pf: 1st assume no zeros or poles on ∂P_0.

Then argument principle $\Rightarrow$

$$2\pi i (N_z - N_p) = \int_{\partial P_0} \frac{f'(z)}{f(z)}\, dz$$

where N_z = number of zeros (counting multiplicity)

N_p = number of poles = order of $f = m$

Same argument as in the proof in Thm 1.3, periodicity $\Rightarrow$

$$\int_{\partial P_0} \frac{f'}{f} = 0 \Rightarrow N_z = N_p = m.$$

If f has zeros or poles on ∂P_0, then works on ∂P of $P = P_0 + h$ with small $h \in \mathbb{C}$. ※

Cor If f is an elliptic function of order m, then $\forall c \in \mathbb{C}$

$f(z) = c$ has m solutions in P_0 (counting multiplicity)

Pf: $\forall c \in \mathbb{C}$, $f - c$ is also an elliptic function with same periods 1 & τ. ※

1.2 The Weierstrass $\wp$ function

(An elliptic function of order 2)

> **Lemma 1.5** If $r > 2$,
>
> $$\sum_{(n,m)\neq(0,0)} \frac{1}{(|n|+|m|)^r} \quad \text{and} \quad \sum_{(n,m)\neq(0,0)} \frac{1}{|n+m\tau|^r} \quad \text{converge}$$

Pf: As both series has positive term, convergent is equivalent to absolute convergent.

Hence the order of sum, i.e. rearrangement, doesn't matter.

(provided we can prove convergence.)

Fix an $n \neq 0$, and consider

$$\sum_{m\in\mathbb{Z}} \frac{1}{(|n|+|m|)^r} = \frac{1}{|n|^r} + 2\sum_{m=1}^{\infty} \frac{1}{(|n|+m)^r}$$

$$= \frac{1}{|n|^r} + 2\sum_{k=|n|+1}^{\infty} \frac{1}{k^r}$$

$$\leq \frac{1}{|n|^r} + 2\int_{|n|}^{\infty} \frac{dx}{x^r}$$

$$= \frac{1}{|n|^r} + \frac{2}{r-1}\frac{1}{|n|^{r-1}}$$

$$\sum_{(n,m)\neq(0,0)} \frac{1}{(|n|+|m|)^r} = \sum_{m\neq 0} \frac{1}{|m|^r} + \sum_{n\neq 0}\sum_{m\in\mathbb{Z}} \frac{1}{(|n|+|m|)^r}$$

$$\leq 2\sum_{m=1}^{\infty} \frac{1}{m^r} + \sum_{n\neq 0}\left(\frac{1}{|n|^r} + \frac{2}{r-1}\,\frac{1}{|n|^{r-1}}\right)$$

$$< +\infty \qquad (\text{since } r>2 \Rightarrow r-1>1)$$

$\therefore$ The series $\displaystyle\sum_{(n,m)\neq(0,0)} \frac{1}{(|n|+|m|)^r}$ converges.

To prove the 2^{nd} series converges, we claim that

<u>Claim</u>: $\forall\ \tau \in \mathbb{H}$, $\exists\ \delta_\tau \in (0,1)$ such that

$$|n+m\tau| \geq \delta_\tau |n+mi| \quad \forall\ n, m \in \mathbb{Z}.$$

<u>Pf of claim</u>: Consider function $f(x) = \dfrac{|x+\tau|}{|x+i|}$, $\forall\ x \in \mathbb{R}$.

Then clearly f is continuous on $\mathbb{R}$, $f(x)>0$, $\forall\ x\in\mathbb{R}$.

and $f(x)\to 1$ as $x\to\pm\infty$. Hence $\exists\ 1>\delta_\tau>0$ s.t. $f(x)\geq\delta_\tau$.

Putting $x=\frac{n}{m}$, $m\neq 0$, we have $\dfrac{|\frac{n}{m}+\tau|}{|\frac{n}{m}+i|} \geq \delta_\tau$

$$\Rightarrow \quad |n+m\tau| \geq \delta_\tau |n+im|, \quad \forall n \text{ and } \forall m\neq 0$$

Since $0<\delta_\tau<1$, the inequality is clearly correct for $m=0$

The claim is proved.

By the claim $|n+m\tau| \geq \delta_\tau |n+mi| = \delta_\tau (n^2+m^2)^{1/2}$

$$\geq \frac{\delta_\tau}{2}(|n|+|m|)$$

Hence $\quad \dfrac{1}{|n+m\tau|^r} \leq \dfrac{2}{\delta_\tau}\dfrac{1}{(|m|+|n|)^r}$

$\therefore$ Convergence of $\displaystyle\sum_{(n,m)\neq(0,0)} \frac{1}{(|m|+|n|)^r}$

$\Rightarrow$ Convergence of $\displaystyle\sum_{(n,m)\neq(0,0)} \frac{1}{|n+m\tau|^r}$ #

<u>Remarks</u>: (i) If $\mathrm{Im}(\tau) \geq t_0 > 0$, then

$$\frac{|x+\tau|}{|x+i|} \geq \frac{|x+s|+t_0}{2|x+i|}, \quad \text{where } s = \mathrm{Re}(\tau).$$

$\therefore$ The lower bound δ_τ can be chosen depending on the lower bound t_0 of $\mathrm{Im}(\tau)$, but not $\mathrm{Im}(\tau)$.

(ii) <u>Eg</u>: Let $\tau = -k+i$ for some $k \in \mathbb{Z}$

Then for $(n,m) = (k,1)$, $|n|+|m| = |k|+1$ and $|n+m\tau| = 1$

$\therefore$ If $\exists\, c>0$ s.t. $c(|n|+|m|) \leq |n+m\tau| \quad \forall (n,m)\in\mathbb{Z}^2$,

we have $\qquad c(|k|+1) \leq 1$

$$\Rightarrow \quad c \leq \frac{1}{|s|+1} \longrightarrow 0 \quad \text{as} \quad |s| \to \infty$$

Hence such constant must depends on the upper bd. of $|\mathrm{Re}(\tau)|$.

($\because$ Remark on Textbook page 269 is not accurate.)

Def: Let Λ = lattice generated by 1 & τ ($\tau \in \mathbb{H}$) and denote

$$\Lambda^* = \Lambda \setminus \{(0,0)\}$$

The <u>Weierstrass $\wp$ Function</u> is defined by

$$\wp = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right]$$

<u>Thm 1.6</u> $\wp$ is an elliptic function with periods 1 and τ, and double poles at the lattic points.

<u>Pf</u>: <u>Step 1</u> The series converges and $\wp$ is a meromaphic function with double poles at $\omega \in \Lambda$.

<u>Pf of Step 1</u>: Let $R > 0$. Then $\forall\ |z| < R$,

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{|\omega| \le 2R \\ \omega \neq (0,0)}} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right] + \sum_{|\omega| > 2R} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right]$$

$$\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} = \frac{\omega^2 - (z+\omega)^2}{(z+\omega)^2 \omega^2} = \frac{-z^2 - 2z\omega}{\omega^2 (z+\omega)^2}$$

$\Rightarrow$ For $|z| < R$ and $|\omega| > 2R$, $\exists\ C > 0$ (depending only on R & τ)

s.t.
$$\left| \frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right| \le \frac{C}{|\omega|^3} = \frac{C}{|n+m\tau|^3} \qquad (|n+m\tau| > 2R)$$

By Lemma 1.5 $\displaystyle\sum_{|\omega| > 2R} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right]$ converges uniformly on $|z| < R$

$\therefore \displaystyle\sum_{|\omega|>2R}\left[\frac{1}{(z+\omega)^2}-\frac{1}{\omega^2}\right]$ is holomorphic in $|z|<R$

$\therefore \wp(z)$ is meromorphic in $\{|z|<R\}$ with double poles

exactly at $z=-\omega\in\Lambda$ with $|\omega|<R$.

Since $R>0$ is arbitrary, Step 1 is proved.

<u>Step 2</u>: $\wp$ has periods 1 and τ.

<u>Pf of Step 2</u>: By the proof of Step 1, $\wp'$ can be calculate termwise

and we have

$$\wp'(z) = -\frac{2}{z^3}+\sum_{\omega\in\Lambda^*}\frac{-2}{(z+\omega)^3} = -2\sum_{n,m\in\mathbb{Z}}\frac{1}{(z+n+m\tau)^3}$$

which converges absolutely (and locally uniformly) for $z\notin\Lambda$.

$$\Rightarrow \begin{cases}\wp'(z+1)=\wp'(z)\\ \wp'(z+\tau)=\wp'(z)\end{cases} \quad \text{for all } z\notin\Lambda$$

($\therefore \wp'$ is an elliptic function of order 3)

Since all poles are triple poles and have no residue,

path integration gives well-defined constants a & b s.t.

$$\begin{cases}\wp(z+1)=\wp(z)+a\\ \wp(z+\tau)=\wp(z)+b\end{cases} \quad \forall z\notin\Lambda$$

Observe that $\wp(z) = \wp(-z)$ by construction.

$$\therefore \quad \begin{cases} \wp(-\frac{1}{2}) = \wp(\frac{1}{2}) = \wp(-\frac{1}{2}+1) = \wp(-\frac{1}{2}) + a \\ \wp(-\frac{\tau}{2}) = \wp(\frac{\tau}{2}) = \wp(-\frac{\tau}{2}+\tau) = \wp(-\frac{\tau}{2}) + b \end{cases}$$

$\Rightarrow a = 0$ and $b = 0$

$\therefore$ $\wp(z)$ has periods 1 & τ.

This completes the proof of the Thm. //

Properties of $\wp$

(1) $\wp'$ is odd

(2) $\wp'(\frac{1}{2}) = \wp'(\frac{\tau}{2}) = \wp'(\frac{1+\tau}{2}) = 0$

and $\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ are the only roots of $\wp'$ in P_0

Pf: (1) $\wp$ even $\Rightarrow$ $\wp'$ odd.

(2) $$\wp'\left(\frac{1+\tau}{2}\right) = -\wp'\left(-\frac{1+\tau}{2}\right) \qquad \text{by (1)}$$

$$= -\wp'\left(-\frac{1+\tau}{2} + 1 + \tau\right) \qquad \text{periods } 1 \,\&\, \tau$$

$$= -\wp'\left(\frac{1+\tau}{2}\right)$$

$$\Rightarrow \wp'\left(\frac{1+\tau}{2}\right) = 0.$$

Similarly, $\wp(\frac{1}{2}) = \wp'(\frac{\tau}{2}) = 0$.

Since $\wp'$ has order 3, Thm 1.4 $\wp'$ has exactly 3 roots in P_0

$\therefore$ $\frac{1}{2}, \frac{\tau}{2}$ & $\frac{1+\tau}{2}$ are all roots of $\wp'$ in P_0 #

Remarks: (i) $\frac{1}{2}, \frac{\tau}{2}$, & $\frac{1+\tau}{2}$ are called the half-periods

(ii) All has multiplicity 1.

Let $e_1 = \wp(\frac{1}{2})$, $e_2 = \wp(\frac{\tau}{2})$, $e_3 = \wp(\frac{1+\tau}{2})$.

Then

(1) $\forall\, i = 1, 2, 3$, $\wp(z) = e_i$ has a double root at

$\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ respectively.

(2) e_i, $i = 1, 2, 3$, are distinct.

Pf: Since $\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ are critical points of $\wp$, they are multiple roots of $\wp(z) = e_i$ in P_0, respectively.

Then $\wp$ has order $2 \Rightarrow$ all are double roots in P_0.

This proves (1).

If e_i are not distinct, then one of $\wp(z) = e_i$ has 4 roots (counting multiplicity) in P_0.

This contradicts the fact that order of $\wp = 2$.

$\therefore$ (2) is proved ※

Thm 1.7 $(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$

Pf: $(\wp')^2$ has double roots at $\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ in P_0 and same for $(\wp - e_1)(\wp - e_2)(\wp - e_3)$.

$(\wp')^2$ has poles of order 6 at $\omega \in \Lambda$ and same for $(\wp - e_1)(\wp - e_2)(\wp - e_3)$.

$\therefore \quad \dfrac{(\wp')^2}{(\wp - e_1)(\wp - e_2)(\wp - e_3)}$ is holomorphic.

Clearly, it is also doubly periodic. Thm 1.2 $\Rightarrow$

$$\frac{(\wp')^2}{(\wp - e_1)(\wp - e_2)(\wp - e_3)} = c \qquad (\text{constant})$$

By the series expansions of $\wp$ and $\wp'$, we have,

near $z=0$, $\quad \begin{cases} \wp(z) = \frac{1}{z^2} + \cdots \\ \wp'(z) = \frac{-2}{z^3} + \cdots \end{cases}$

$\therefore$ Near $z=0$,

$$c = \frac{(\wp')^2}{(\wp - e_1)(\wp - e_2)(\wp - e_3)} = \frac{\frac{4}{z^6}(1+\cdots)}{\frac{1}{z^6}(1+\cdots)}$$

$\Rightarrow \quad c = 4$. #