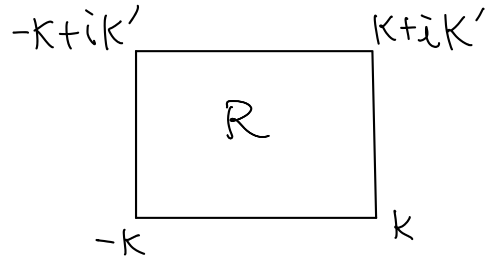
4.5 Return to Elliptic Integrals

In Eg 3 of section 4.1, it was shown that the elliptic integral

$$I(z) = \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} \qquad (0<k<1)$$

maps $\mathbb{R}$-axis to the boundary of the rectangle:



where $\displaystyle K = K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$

$$K' = K'(k) = \int_1^{1/k} \frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}}.$$

However, we mentioned that we haven't proved that $I(z)$ maps $\mathbb{H}$ conformally onto R. Now, we can show it by Thm 4.6 as follows.

For this purpose, we need a lemma

Lemma (Ex 15 of the Textbook on pg 251)

(a) If $\Phi \in \mathrm{Aut}(\mathbb{H})$ and $\exists$ distinct points A_1, A_2, A_3 on $\mathbb{R}$-axis such that

$$\Phi(A_i) = A_i, \quad \text{for } i = 1, 2, 3.$$

Then $\Phi = \mathrm{Id}_{\mathbb{H}}$

(b) Let $x_1 < x_2 < x_3$ and $y_1 < y_2 < y_3$ $(x_i, y_i \in \mathbb{R})$.

Then $\exists\ \Phi \in \mathrm{Aut}(\mathbb{H})$ such that

$$\Phi(x_i) = y_i, \quad i = 1, 2, 3.$$

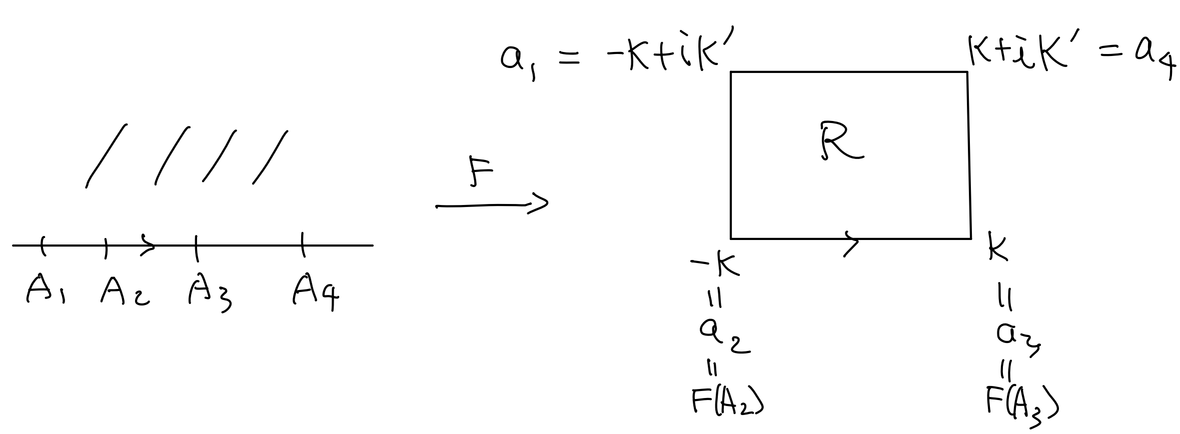
Same conclusion holds if $y_3 < y_1 < y_2$ or $y_2 < y_3 < y_1$

(Pf: Ex!)

<u>Pf of $I(z) : \mathbb{H} \to R$ conformal</u>

Let $F : \mathbb{H} \to R$ be a conformal map (existence by Riemann Mapping)

Let $A_1 < A_2 < A_3 < A_4$ (A_4 may $= \infty$) be points that maps to the vertices $-K, K, K+iK', -K+iK'$



By Thm 4.6, F maps $[A_2, A_3]$ to $[-K, K]$.

Hence $$A_2 < F^{-1}(0) < A_3$$

By lemma above, $\exists\ \Phi \in \mathrm{Aut}(\mathbb{H})$ such that

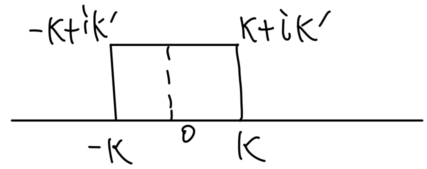
$$\Phi(-1) = A_2,\ \Phi(0) = F^{-1}(0),\ \Phi(1) = A_3.$$

$\Rightarrow G = F \circ \Phi : \mathbb{H} \to R$ conformal and satisfies

$$\begin{cases} G(-1) = -K \\ G(0) = 0 \\ G(1) = K \end{cases}.$$

Then note that the upper-half plane $\mathbb{H}$ and the rectangle R are symmetric wrt

$$x + iy = z \mapsto -\bar{z} = -x + iy$$



$$G^*(z) = -\overline{G(-\bar{z})} : \mathbb{H} \xrightarrow{z \mapsto -\bar{z}} \mathbb{H} \xrightarrow{G} R \xrightarrow{w \mapsto -\bar{w}} R$$

Cauchy-Riemann equation (& Chain rule) (Typo in the Textbook.)

$\Rightarrow G^* : \mathbb{H} \to R$ is also conformal.

Hence $G^{-1} \circ G^* : \mathbb{H} \to \mathbb{H} \in \mathrm{Aut}(\mathbb{H})$.

Observe $G^*(1) = -\overline{G(-1)} = K = G(1)$

$$G^*(-1) = -\overline{G(1)} = -K = G(-1)$$

$$G^*(0) = -\overline{G(0)} = 0 = G(0).$$

The Lemma $\Rightarrow$ $G^{-1}\circ G^* = \mathrm{Id}_{\mathbb{H}}$, i.e. $G = G^*$.

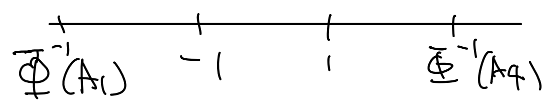
$$K + iK' = F(A_4) = G(\Phi^{-1}(A_4)) = G^*(\Phi^{-1}(A_4)) = -\overline{G(-\overline{\Phi^{-1}(A_4)})}$$

$$\Rightarrow \quad G(-\Phi^{-1}(A_4)) = \overline{(-K-iK')} = -K+iK' = F(A_1) = G(\Phi^{-1}(A_1))$$

By injectivity, $\Phi^{-1}(A_1) = -\Phi^{-1}(A_4)$.

Together with the orientation, we must have

$$\Phi^{-1}(A_4) > 1$$



And hence $\exists\, l \in (0,1)$ s.t. $\Phi^{-1}(A_4) = \frac{1}{l}$.

Altogether, we may assume the map

$F : \mathbb{H} \to P$ and points A_1, A_2, A_3, A_4 at the beginning of the proof satisfies

$$F(0) = 0 \quad \text{and}$$

$$A_1 = -\frac{1}{l}, \quad A_2 = -1, \quad A_3 = 1, \quad A_4 = \frac{1}{l} \quad (0 < l < 1)$$

By Thm 4.6, $\exists\, C_1, C_2$ such that

$$F(z) = C_1 \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-l^2\zeta^2)}} + C_2$$

(Note that more precisely, $c_1' \int_0^z \frac{d\zeta}{\sqrt{(\zeta+\frac{1}{l})(\zeta+1)(\zeta-1)(\zeta-\frac{1}{l})}} + c_2$

$\therefore\ c_1 = l c_1'$)

Using $F(0) = 0$, we have $C_2 = 0$

Putting $z = 1, \frac{1}{\ell}$ in the formula, we have

$$K(k) = K = c_1 \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-\ell^2\xi^2)}} = c_1 K(\ell)$$

and

$$K + iK' = F\left(\frac{1}{\ell}\right) = c_1 \left[K(\ell) + \int_1^{\frac{1}{\ell}} \frac{d\xi}{\sqrt{(1-\xi^2)(1-\ell^2\xi^2)}} \right]$$

$$= c_1 K(\ell) + c_1 i \int_1^{\frac{1}{\ell}} \frac{dx}{\sqrt{(x^2-1)(1-\ell^2 x^2)}}$$

$$\Rightarrow \quad K'(k) = c_1 K'(\ell)$$

By Ex. 24 of the Textbook,

$$K'(k) = K(\sqrt{1-k^2})$$

$\therefore$ we have
$$\begin{cases} K(k) = c_1 K(\ell) \\ K(\sqrt{1-k^2}) = c_1 K(\sqrt{1-\ell^2}) \end{cases}$$

$$\Rightarrow \quad \frac{K(k)}{K(\sqrt{1-k^2})} = \frac{K(\ell)}{K(\sqrt{1-\ell^2})}$$

Clearly $K(k)$ is strictly increasing in k, $(0<k<1)$ (Ex!)

$\Rightarrow$ $K(\sqrt{1-k^2})$ is strictly decreasing in k.

$\therefore$ $\dfrac{K(k)}{K(\sqrt{1-k^2})}$ is strictly increasing in k.

Hence $k = \ell$, and then $c_1 = 1$.

$$\therefore \quad F(z) = \int_0^z \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} \qquad \text{※}$$