Lemma 4.4 Let $z_0 \in \partial\mathbb{D}$. Then $\displaystyle\lim_{\substack{z\to z_0\\ z\in\mathbb{D}}} F(z)$ exists.

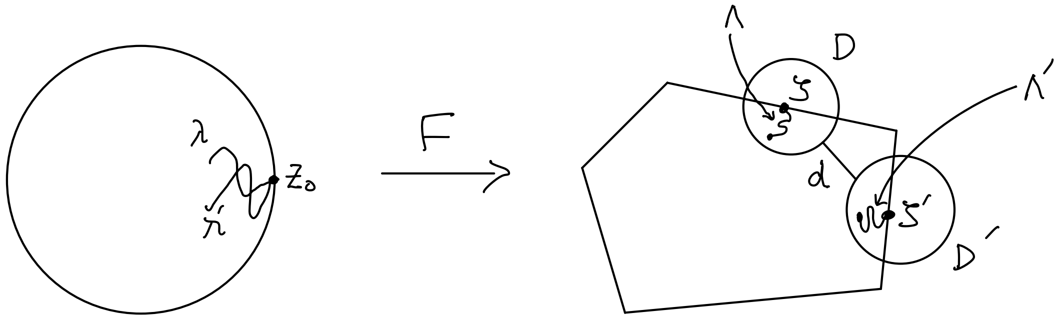
Pf: Suppose not.

Then $\exists$ two sequences $\{z_n\}$ and $\{z_n'\}$ s.t.

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} z_n' = z_0 \quad \text{and}$$

$$\lim_{n\to\infty} F(z_n) = \zeta \neq \zeta' = \lim_{n\to\infty} F(z_n')$$

Since $F: \mathbb{D}\to P$ is conformal, $\zeta, \zeta' \in \partial P = \bar{P}$.



Let D, D' be discs centered at ζ & ζ' respective with $d = \text{dist}(D, D') > 0$.

Then $\exists\, n_0 > 0$ s.t. if $n \geq n_0$, $F(z_n) \in D\cap P$ &

$F(z_n') \in D'\cap P$

$\Rightarrow$ $\exists$ a continuous curve Λ in $D\cap P$ connecting all $F(z_n)$ with $n \geq n_0$ & ζ

i.e. $F(z_n) \in \Lambda$, $\forall\, n \geq n_0$ and

ζ is one of the end point of Λ

Similarly for Λ' in $D'\cap P$

Let $\lambda = F^{-1}(\Lambda)$ and $\lambda' = F^{-1}(\Lambda')$

Then λ, λ' are continuous curves in $\mathbb{D}$

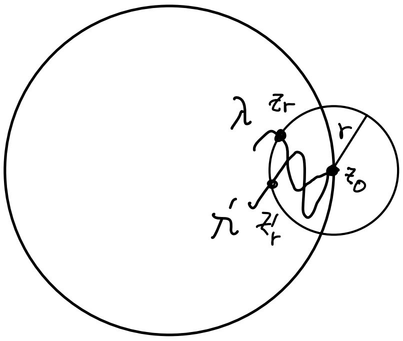
and $z_n \in \lambda$, $z_n' \in \lambda'$, $\forall\, n \geq n_0$

Since $z_n \to z_0$ & $z_n' \to z_0$,

continuity of λ & λ' implies

C_r intersects λ and λ' at some

point z_r and z_r' respectively. (C_r as in Lemma 4.3)



Lemma 4.3 $\Rightarrow$ $\exists\, r_n \to 0$ s.t.

$$\rho(r_n) = |F(z_{r_n}) - F(z'_{r_n})| \to 0 \quad \text{as } n \to \infty.$$

This is a contradiction

since
$$\begin{cases} z_{r_n} \in \lambda \Rightarrow F(z_{r_n}) \in \Lambda \in D \cap P \\ z'_{r_n} \in \lambda' \Rightarrow F(z'_{r_n}) \in \Lambda \in D' \cap P \end{cases}$$

$$\Rightarrow \quad |F(z_{r_n}) - F(z'_{r_n})| \geq \operatorname{dist}(D, D') > 0.$$

$\therefore$ $\displaystyle\lim_{\substack{z \to z_0 \\ z \in D}} F(z)$ exists ※

Lemma 4.5 The conformal map $F : \mathbb{D} \to \mathbb{P}$ extends to a continuous map from $\overline{\mathbb{D}}$ to $\overline{\mathbb{P}}$.

Pf: For $z_0 \in \partial \mathbb{D}$, define

$$F(z_0) = \lim_{\substack{z \to z_0 \\ z \in \mathbb{D}}} F(z).$$

Existence of the limit is proved in Lemma 4.4.

Clearly, it remains to show that F is continuous at $z_0 \in \partial\mathbb{D}$.

By definition of $F(z_0)$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$(*) \qquad |F(z) - F(z_0)| < \varepsilon, \text{ if } |z - z_0| < \delta \text{ and } z \in \mathbb{D}.$$

For $z \in \partial\mathbb{D}$, and $|z - z_0| < \delta$

then $\exists\, w \in \mathbb{D}$, close to z, such that $|F(w) - F(z)| < \varepsilon$ and

$$|w - z_0| < \delta.$$

Therefore
$$\begin{aligned}|F(z) - F(z_0)| &\le |F(z) - F(w)| + |F(w) - F(z_0)| \\ &< \varepsilon + \varepsilon \qquad (\text{applying } (*) \text{ to } w) \\ &= 2\varepsilon.\end{aligned}$$

All together F is continuous on $\partial\mathbb{D}$. #

Pf of Thm 4.2

Consider $G = F^{-1} : P \to \mathbb{D}$.

Observe that the same argument $\Rightarrow$ G also extends to a continuous map from $\overline{P}$ to $\overline{\mathbb{D}}$.

It is clear that $F(\partial\mathbb{D}) \subset \mathcal{P} = \partial P$ and $G(\partial P) \subset \partial\mathbb{D}$.

If $z \in \partial\mathbb{D}$, take a seq. $\{z_k\} \subset \mathbb{D}$ s.t. $z_k \to z$.

Then $G(F(z_k)) = z_k$, $\forall k$

Taking limit as $k \to \infty$ and using the fact that both F & G extends continuously to the boundary, we have

$$G(F(z)) = z$$

Similarly for $w \in \partial P = \mathcal{P}$, $F(G(w)) = w$.

This completes the proof of the Thm. ※

4.4 The Mapping Formula

Let
- P = bounded polygonal region
- $\wp$ = boundary polygon of P
- $a_1, a_2, \cdots, a_n$ ordered vertices of $\wp$ $(n \geq 3)$.
- $\pi \alpha_k$ = interior angle at a_k.
- $\pi \beta_k$ = exterior angle at a_k, i.e. $\beta_k = 1 - \alpha_k$

Then $\sum_{k=1}^{n} \beta_k = 2$ (Elementary Euclidean Geometry)

Let $F : \mathbb{H} \to P$ be conformal

- Existence is guaranteed by Riemann mapping thm :

$$\begin{tikzcd}
\mathbb{H} \arrow[r] \arrow[rr, bend left, "F"] & \mathbb{D} \arrow[r] & P \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \\
z \arrow[r, mapsto] & w = \frac{i-z}{i+z} \arrow[r, mapsto, "\uparrow \text{ Riemann map}"'] & G(w) = F(z)
\end{tikzcd}$$

- Since G extends continuously to $\overline{\mathbb{D}}$ by Thm 4.2 and

  $z \mapsto w = \frac{i-z}{i+z}$ clearly extends continuously

  to the boundary x-axis,

  the conformal map $F : \mathbb{H} \to P$ extends continuously to $\overline{\mathbb{H}}$.

- May assume $A_k = F^{-1}(a_k) \in \mathbb{R}$ (i.e. no vertex of $\wp \leftrightarrow \infty$)
  (Ex!)

- F continuous & bijective

  $\Rightarrow$ $A_1 < \cdots < A_n$ (by relabeling a_k if needed)

- Then $[A_k, A_{k+1}]$ $(1 \leq k \leq n-1)$ $\xrightarrow{F}$ $[a_k, a_{k+1}]$

  $(-\infty, A_1] \cup [A_n, \infty)$ $\xrightarrow{F}$ $[a_n, a_1]$

Thm 4.6 Let $F : \mathbb{H} \to P$ conformal, s.t. $F(\infty)$ is not a vertex of P.

S = Schwarz-Christoffel integral in subsection 4.2

with A_k & β_k as above

Then $\exists$ (cpx) constants C_1 and C_2 such that

$$F(z) = C_1 S(z) + C_2 . \qquad (C_1 \neq 0)$$

Idea of proof: If $F = C_1 S + C_2$,

then
$$F'(z) = \frac{C_1}{(z-A_1)^{\beta_1} \cdots (z-A_n)^{\beta_n}}$$

$$\Rightarrow \log F'(z) = \log C_1 - \sum_{k=1}^{n} \beta_k \log(z - A_k) \quad \left(\text{whenever defined}\right)$$

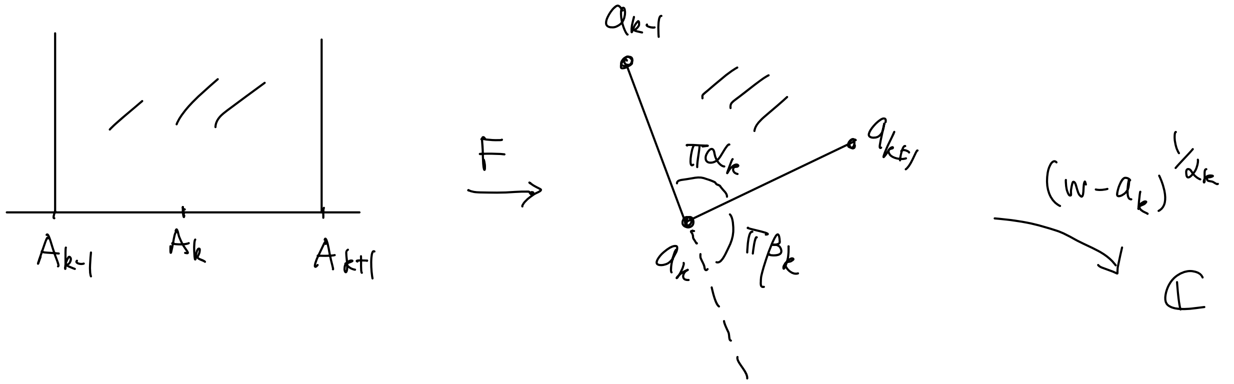
$$\Rightarrow \frac{F''(z)}{F'(z)} + \sum_{k=1}^{n} \frac{\beta_k}{z-A_k} = 0 .$$

Hence we need to study: $\begin{cases} \text{(i) behavior of } F \text{ at } A_k \\ \text{(ii) behavior of } F \text{ at } \infty . \end{cases}$

Hope that (i) it gives the correct singularities at A_k &

(ii) to conclude it is the constant zero.

Pf. Step 1 Local behavior of F''/F' at $z=A_k$.



Consider $h_k(z) = \left[F(z)-a_k\right]^{1/\alpha_k}$ for $k=2,\cdots,n-1$

Since $\{ A_{k-1} < x < A_{k+1},\ y>0 \}$ is simply-connected

and $F(z) \neq a_k$ ($\forall z$ inside),

$h_k(z)$ is well-defined by choosing a branch of log.

By $1/\alpha_k > 0$ and $F(z)$ continuous up to boundary including $z=A_k$,

we see that $h_k(z)$ extends to the horizontal

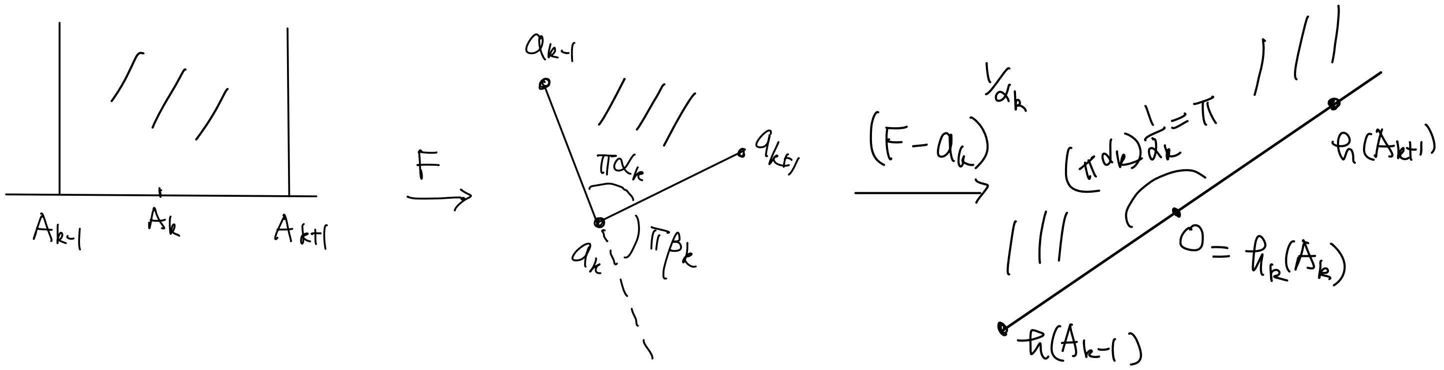
line segment (A_{k-1}, A_{k+1}).

Note that the value of the extended $h_k(z)$ at $z=A_k$

is $h_k(A_k)=0$ as

$$|h_k(z)| = |F(z)-a_k|^{1/\alpha_k} \to 0 \text{ as } z \to A_k$$

(which shows the continuity at $z=A_k$ too.)

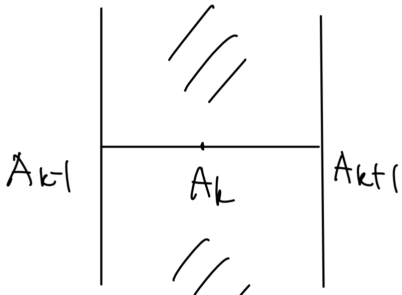


And the image of the horizontal arc (A_{k-1}, A_{k+1}) is a straight line segment by the property of the map $z \mapsto z^{1/\alpha_k}$.

Hence composing with a rotation of a suitable angle θ_k, $e^{i\theta_k} h_k$ maps (A_{k-1}, A_{k+1}) into $\mathbb{R}$.

Schwarz reflection principle $\Rightarrow$ $e^{i\theta_k} h_k$ and hence h_k can be analytically continuated to the infinite strip

$\{z = x+iy : A_{k-1} < x < A_{k+1}\}$.



<u>Claim</u>: $h_k'(z) \neq 0$, $\forall z \in \{A_{k-1} < \mathrm{Re}(z) < A_{k+1}\}$

Note that for $z \in \{A_{k-1} < \mathrm{Re}(z) < A_{k+1}\} \cap \mathbb{H}$

$$\frac{h_k'(z)}{h_k(z)} = \frac{1}{\alpha_k} \frac{F'(z)}{F(z) - a_k}$$

$F : \mathbb{H} \to P$ is conformal $\Rightarrow$ $F'(z) \neq 0$, $\forall z \in \mathbb{H}$.

Hence $h_k'(z) \neq 0$, $\forall$ $z \in \{A_{k-1} < Re(z) < A_{k+1}\} \cap \mathbb{H}$.

Recall that, the Schwarz reflection principle is constructed using $\overline{e^{i\theta_k} h_k(\bar{z})}$. So up to a multiple of nonzero constant, $h_k'(z)$ for $z \in \{A_{k-1} < Re(z) < A_{k+1}, \ Im\, z < 0\}$ is $\overline{h_k'(\bar{z})}$ and hence $h_k'(z) \neq 0$.

It remains to consider $z = x \in (A_{k-1}, A_{k+1})$.

Note that $F\big|_{(A_{k-1}, A_{k+1})}$ is injective,

$h_k\big|_{(A_{k-1}, A_{k+1})}$ is also injective.

Hence $\frac{\partial}{\partial x} h_k(x) \neq 0$ $\forall$ $x \in (A_{k-1}, A_{k+1})$. ($\swarrow$ cpx derivative)

Since h_k is holo in a nbd of $z = x$, $h_k'(x) \neq 0$.

We've shown that $h_k'(z) \neq 0$, $\forall z \in \{A_{k-1} < Re(z) < A_{k+1}\}$.

For $z \in \{A_{k-1} < Re(z) < A_{k+1}\} \cap \mathbb{H}$,

$$F'(z) = \alpha_k \cdot (F(z) - a_k) \cdot \frac{h_k'(z)}{h_k(z)}$$

$$= \alpha_k\, h_k(z)^{\alpha_k - 1}\, h_k'(z)$$

$$= \alpha_k\, h_k(z)^{-\beta_k}\, h_k'(z)$$

$$\Rightarrow \quad \frac{F''(z)}{F'(z)} = -\beta_k \frac{h_k'(z)}{h_k(z)} + \frac{h_k''(z)}{h_k'(z)}$$

Since $h_k'(z) \neq 0$, $\frac{F''(z)}{F'(z)}$ extended to a meromorphic function in $\{A_{k-1} < Re(z) < A_{k+1}\}$ with a simple pole at the only zero $z = A_k$ of $h_k(z)$.

(clear from the definition of h_k on the upper half strip and the reflection principle)

By Taylor's expansion of $h_k(z)$ near $z = A_k$, the

$$\operatorname{Res}_{z=A_k} \frac{F''(z)}{F'(z)} = -\beta_k .$$

$$\therefore \frac{F''(z)}{F'(z)} + \sum_{l=1}^{n} \frac{\beta_l}{z-A_l} = \left[\beta_k \cdot \left(\frac{1}{z-A_k} - \frac{h_k'(z)}{h_k(z)}\right) + \frac{h_k''(z)}{h_k'(z)}\right] + \sum_{l \neq k} \frac{\beta_l}{z-A_l}$$

$$= E_k(z)$$

with $E_k(z)$ is holo in $\{A_{k-1} < Re(z) < A_{k+1}\}$

(as $\frac{1}{z-A_l}$, $l \neq k$, are also holo in the domain)

Similarly, there exists $E_1(z)$ holo in $\{-\infty < Re(z) < A_2\}$

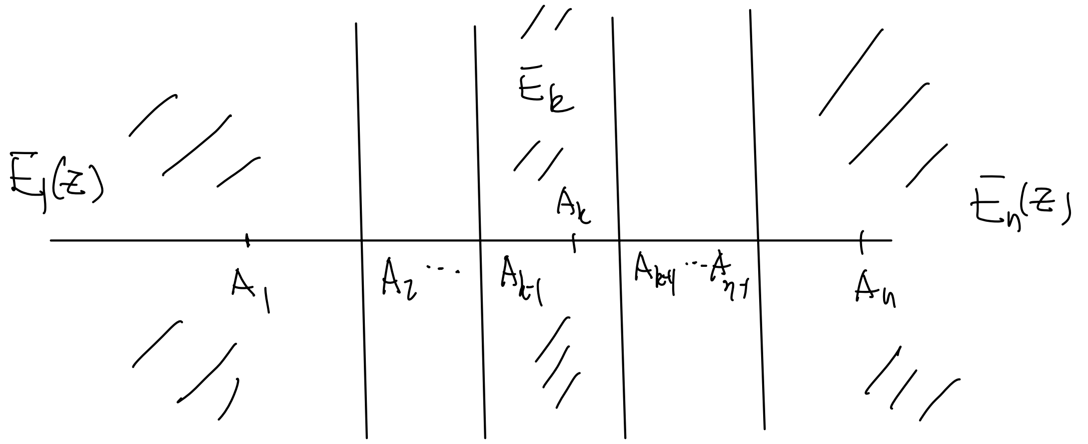
such that

$$\frac{F''(z)}{F'(z)} + \sum_{l=1}^{n} \frac{\beta_l}{z-A_l} = E_1(z), \quad \forall z \in \{-\infty < Re(z) < A_2\},$$

and $E_n(z)$ holo in $\{A_{n-1} < Re(z) < \infty\}$

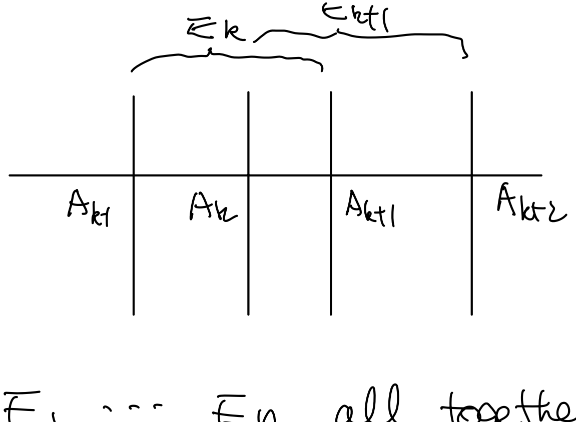
such that

$$\frac{F''(z)}{F'(z)} + \sum_{l=1}^{n} \frac{\beta_l}{z - A_l} = E_n(z), \quad \forall z \in \{A_{n-1} < Re(z) < \infty\},$$



Step 2 Global behavior of $F''(z)/F'(z)$

Note that the domains of E_k & E_{k+1} overlaps on $\{A_k < Re(z) < A_{k+1}\}$, E_k & E_{k+1} together define an analytic function on $\{A_{k-1} < Re(z) < A_{k+2}\}$. (and agree on $\mathbb{H}$)



And so on, $E_1, \cdots, E_n$ all together defines an entire function $E(z)$ (on $\{-\infty < Re(z) < +\infty\}$).

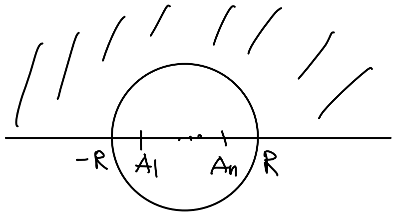
This implies, $\frac{F''(z)}{F'(z)} + \sum_{k=1}^{n} \frac{\beta_k}{z-A_k}$ is entire

(More precisely, extends to an entire function.)

Step 3: Estimate of $F''(z)/F'(z)$ at $|z| \to \infty$.

Note that by uniqueness of analytic continuation, the extension of $\frac{F''(z)}{F'(z)}$ given above is equal to the extension given below:

Consider $R > \max\{|A_1|, |A_n|\}$, then $F(z)$ is holomorphic on $\mathbb{H} \cap \{|z| > R\}$.



Then Thm 4.2 (and similar argument as in Prop 4.1 (i))

$\Rightarrow$ $F(z)$ maps $(-\infty, -R) \cup (R, +\infty)$

into the straight line segment (a_n, a_1) (an edge of $\mathfrak{p}$).

Hence one can apply Schwarz reflect principle as before to extend $F(z)$ analytically to

$$\{z : |z| > R\}.$$

Moreover, similar argument as in the proof of $h_k'(z) \neq 0$ before,

we have F conformal $\Rightarrow$ $F'(z) \neq 0$ $\forall z \in \{|z| > R\}$.

And hence $F''(z)/F'(z)$ is well-defined on $\{|z|>R\}$

Since this extension coincides with the previous one on $\{|z|>R\}\cap\mathbb{H}$, they are identical.

Now, by Laurent expansion on $\{|z|>R\}$

$$F(z) = C_0 + \frac{C_k}{z^k} + \frac{C_{k+1}}{z^{k+1}} + \cdots \quad \text{with } C_k \neq 0, \quad k \geq 1$$

and $C_0 = F(\infty) \in \mathbb{C}$.

$$\Rightarrow \begin{cases} F'(z) = -\frac{kC_k}{z^{k+1}} - \frac{(k+1)C_{k+1}}{z^{k+2}} + \cdots \\ F''(z) = \frac{k(k+1)C_k}{z^{k+2}} + \frac{(k+1)(k+2)C_{k+1}}{z^{k+3}} + \cdots \end{cases}$$

$$\Rightarrow \quad \frac{F''(z)}{F'(z)} = -\frac{(k+1)}{z} \cdot \frac{\left(1 + \frac{k+2}{k} \cdot \frac{C_{k+1}}{C_k} \cdot \frac{1}{z} + \cdots\right)}{\left(1 + \frac{k+1}{k} \cdot \frac{C_{k+1}}{C_k} \cdot \frac{1}{z} + \cdots\right)} \quad \text{for } \{|z|>R\}$$

$$\Rightarrow \quad \frac{|F''(z)|}{|F'(z)|} \to 0 \quad \text{as } |z| \to \infty.$$

Final Step:

By Steps 2 & 3, the entire function

$$\frac{F''(z)}{F'(z)} + \sum_{k=1}^{n} \frac{\beta_k}{z - A_k} \to 0 \quad \text{as } |z| \to \infty.$$

Liouville's Thm $\Rightarrow$ $\dfrac{F''(z)}{F'(z)} + \sum_{k=1}^{n} \dfrac{\beta_k}{z-A_k} = 0$.

Let $Q(z) = \dfrac{1}{(z-A_1)^{\beta_1} \cdots (z-A_n)^{\beta_n}} = S'(z)$

Then $\forall z \in \mathbb{H}$,

$$\frac{d}{dz}\left(\frac{F'(z)}{Q(z)}\right) = \frac{F'(z)}{Q(z)}\left[\frac{F''(z)}{F'(z)} + \sum_{k=1}^{n} \frac{\beta_k}{z-A_k}\right] = 0$$

$\Rightarrow$ $F'(z) = C_1 Q(z)$ $\forall z \in \mathbb{H}$, for some constant $\overset{\neq 0}{C_1}$

Hence $F(z) = C_1 S(z) + C_2$, $\forall z \in \mathbb{H}$, for some const. C_2. #

Note that in Thm 4.6, $F(\infty)$ <u>can't</u> be a vertex of $\mathbb{P}$.
If $F(\infty)$ is a vertex of $\mathbb{P}$, then the formula actually simpler with only $(n-1)$-degree in the denominator.

With the same notations as before, we have

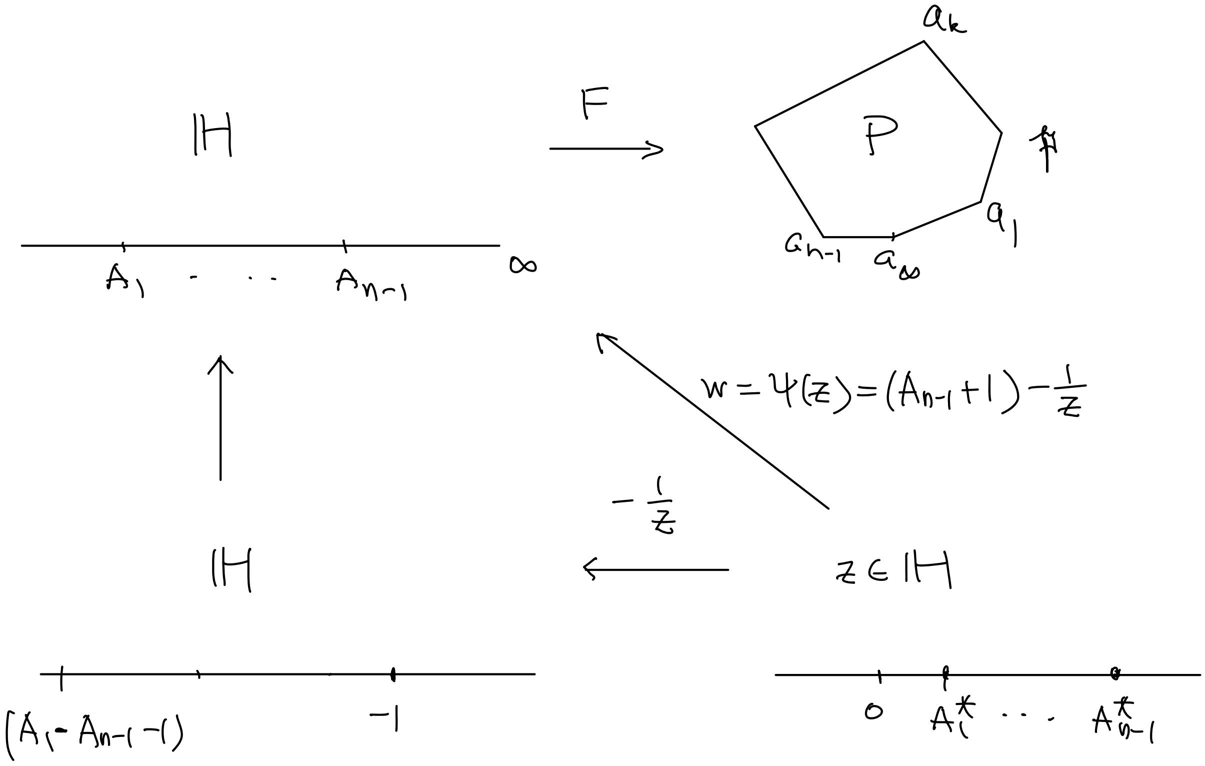
Thm 4.7 If $F : \mathbb{H} \to P$ conformal and

maps $A_1 < \cdots < A_{n-1}, \infty$ to the vertices of $\mathfrak{P}$,

then $\exists$ (cpx) constants C_1 and C_2 such that

$$F(z) = C_1 \int_0^z \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\zeta - A_{n-1})^{\beta_{n-1}}} + C_2$$

Pf:



Define $\psi(z) = (A_{n-1}+1) - \frac{1}{z} = \frac{(A_{n-1}+1)z + (-1)}{1 \cdot z + 0}$

with $A_{n-1}+1, -1, 1, 0 \in \mathbb{R}$ and $(A_{n-1}+1)\cdot 0 - (-1)\cdot 1 = 1$

$\therefore$ $\psi \in \mathrm{Aut}(\mathbb{H})$.

Moreover $z = \psi^{-1}(w) = \frac{1}{(A_{n-1}+1) - w}$

Let $A_k^* = \Psi^{-1}(A_k) = \dfrac{1}{(A_{n-1}+1)-A_k} = \dfrac{1}{(A_{n-1}-A_k)+1} > 0$ and $\neq \infty$.

since $(A_{n-1}-A_k)+1 \geq 1$, $\forall\ k=1,\cdots,n-1$.

Also

$$A_k^* - A_{k-1}^* = \frac{1}{(A_{n-1}+1)-A_k} - \frac{1}{(A_{n-1}+1)-A_{k-1}}$$

$$= \frac{A_k - A_{k-1}}{(A_{n-1}-A_k+1)(A_{n-1}-A_{k-1}+1)} > 0$$

For $w=\infty$, we have

$$0 = \Psi^{-1}(\infty)$$

Hence $F \circ \Psi : \mathbb{H} \to \mathbb{P}$ conformal and

maps $0 < A_1^* < \cdots < A_{n-1}^*$ to $a_\infty, a_1, \cdots, a_{n-1}$ of $\mathbb{P}$

(still in order)

with exterior angles $\beta_\infty, \beta_1, \cdots, \beta_{n-1}$ satisfying

$$\beta_\infty + \sum_{k=1}^{n-1} \beta_k = 2 .$$

Applying Thm 4.6, $\exists$ constants C_1' & C_2' such that

$$F \circ \Psi(z) = C_1' \int_0^z \frac{d\zeta}{\zeta^{\beta_\infty}(\zeta - A_1^*)^{\beta_1} \cdots (\zeta - A_{n-1}^*)^{\beta_{n-1}}} + C_2'$$

$$\Rightarrow \quad F(w) = C_1' \int_0^{\psi^{-1}(w)} \frac{d\zeta}{\zeta^{\beta_\infty}(\zeta - A_1^*)^{\beta_1} \cdots (\zeta - A_{n-1}^*)^{\beta_{n-1}}} + c_2'$$

$$= C_1' \left(\int_{\psi^{-1}(0)}^{\psi^{-1}(w)} + \int_0^{\psi^{-1}(0)} \right) \frac{d\zeta}{\zeta^{\beta_\infty}(\zeta - A_1^*)^{\beta_1} \cdots (\zeta - A_{n-1}^*)^{\beta_{n-1}}} + c_2'$$

Note that $\psi^{-1}(0) = \infty$.

Since the integral converges, the famula is valid.

(in fact, $C_1' \int_0^{\psi^{-1}(0)} \frac{d\zeta}{\zeta^{\beta_\infty} \prod_{k=1}^{n-1} (\zeta - A_k^*)^{\beta_k}} + c_2'$ converges to a point on $\mathfrak{P}$)

$$\therefore \quad F(w) = C_1' \int_\infty^{\psi^{-1}(w)} \frac{d\zeta}{\zeta^{\beta_\infty}(\zeta - A_1^*)^{\beta_1} \cdots (\zeta - A_{n-1}^*)^{\beta_{n-1}}} + C_2$$

where $C_2 = C_1' \int_0^{\psi^{-1}(0)} \frac{d\zeta}{\zeta^{\beta_\infty} \prod_{k=1}^{n-1} (\zeta - A_k^*)^{\beta_k}} + c_2'$

Now substitute $\varphi = \psi(\zeta) = (A_{n-1} + 1) - \frac{1}{\zeta}$

Then $\zeta = \frac{1}{A_{n-1} + 1 - \varphi}$ and $d\zeta = \frac{d\varphi}{(A_{n-1} + 1 - \varphi)^2}$.

$$\Rightarrow F(w) = C_2 + C_1' \int_0^w \frac{1}{\psi^{-1}(\varphi)^{\beta_\infty} \prod_{k=1}^{n-1} (\psi^{-1}(\varphi) - A_k^*)^{\beta_k}} \circ \frac{d\varphi}{(A_{n-1} + 1 - \varphi)^2}$$

Note that $\Psi^{-1}(\varphi) - A_k^* = \Psi^{-1}(\varphi) - \Psi^{-1}(A_k)$

$$= \frac{1}{A_{n-1}+1-\varphi} - \frac{1}{A_{n-1}+1-A_k}$$

$$= \frac{\varphi - A_k}{(A_{n-1}+1-\varphi)(A_{n-1}+1-A_k)}$$

$$\therefore\ F(w) = C_2 + C_1' \int_0^w \frac{(A_{n-1}+1-\varphi)^{\beta_\infty} \prod_{k=1}^{n-1} \left[(A_{n-1}+1-\varphi)(A_{n-1}+1-A_k)\right]^{\beta_k}}{\prod_{k=1}^{n-1} (\varphi - A_k)^{\beta_k} \quad (A_{n-1}+1-\varphi)^2} \, d\varphi$$

Using $2 = \beta_\infty + \sum_{k=1}^{n-1} \beta_k$,

$$F(w) = C_2 + C_1' \cdot \prod_{k=1}^{n-1} (A_{n-1} - A_k + 1)^{\beta_k} \cdot \int_0^w \frac{d\varphi}{\prod_{k=1}^{n-1} (\varphi - A_k)^{\beta_k}}$$

$$= C_1 \int_0^w \frac{d\varphi}{\prod_{k=1}^{n-1} (\varphi - A_k)^{\beta_k}} + C_2,$$

where $C_1 = C_1' \cdot \prod_{k=1}^{n-1} (A_{n-1} - A_k + 1)^{\beta_k}$,

which is the desired formula. ※