4.3 Boundary Behavior

Let P = polygonal region with boundary $\mathbb{P}$ (polygon)

Then P is bounded, simply-connected open & connected.

Thm 4.2 If $F : \mathbb{D} \to P$ is a conformal map,

then F extends to a continuous bijection

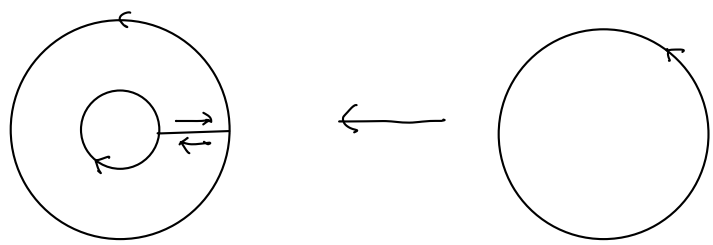
from the closure $\overline{\mathbb{D}}$ to the closure $\overline{P}$.

In particular $F|_{\partial\mathbb{D}} : \partial\mathbb{D} \to \mathbb{P}$ cts. & bijective.

Remark: Thm 4.2 is not true for general proper simply-connected regions.

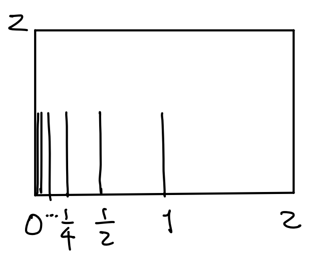
It is true $\iff$ $\partial\Omega$ is a Jordan curve.

Eg 1: $\Omega =$



Boundary map cannot be injective. (Proof Omitted)

Eg 2: $\Omega =$



$$(0,2)\times(0,2) \setminus \bigcup_{n=1}^{\infty} \{ \tfrac{1}{n} + iy : 0 < y \le 1 \}$$

is a simply-connected proper region. But $F : \mathbb{D} \to \Omega$ cannot be extended continuously to $\partial\mathbb{D}$. (Proof omitted)

Pf of Thm 4.2

Recall that the Jacobian determinant of a holo function F is $|F'(z)|^2$ when regarded as a 2-variables to 2-variables transformation $w = f(z)$.

Hence for conformal $F : U \to F(U)$

$$\text{Area } F(U) = \iint_U |F'(z)|^2 \, dx\,dy$$

Lemma 4.3 Let $z_0 \in \partial\mathbb{D}$, and

$$C_r = \{\zeta : |\zeta - z_0| = r\}, \quad \forall\, 0 < r < \tfrac{1}{2}$$

Suppose that for sufficiently small r,

two points $z_r, z_r' \in \mathbb{D} \cap C_r$ are given, and denote

$$\rho(r) = |F(z_r) - F(z_r')|.$$

Then $\exists$ seq. r_n with $r_n \to 0$ as $n \to +\infty$ such that

$$\lim_{n\to\infty} \rho(r_n) = 0$$

Pf: Suppose not.

Then $\exists\, c > 0$ and $0 < R < \tfrac{1}{2}$ such that

$$\rho(r) \geq c, \quad \forall\, 0 < r \leq R$$

Let α be the arc on C_r joining z_r & z_r' in $\mathbb{D}$,

then $$F(z_r) - F(z_r') = \int_\alpha F'(\zeta)\, d\zeta$$

Parametrize α by $\zeta = z_0 + re^{i\theta}$, $\theta_1(r) \le \theta \le \theta_2(r)$,

then $\rho(r) \le \int_{\theta_1(r)}^{\theta_2(r)} |F'(\zeta)| \, r \, d\theta$

(Cauchy-Schwarz) $\le \left(\int_{\theta_1(r)}^{\theta_2(r)} |F'(\zeta)|^2 r \, d\theta \right)^{1/2} \left(\int_{\theta_1(r)}^{\theta_2(r)} r \, d\theta \right)^{1/2}$

($\uparrow \le 2\pi r$)

$\Rightarrow$

$$\frac{c^2}{r} \le \frac{\rho^2(r)}{r} \le 2\pi \int_{\theta_1(r)}^{\theta_2(r)} |F'(\zeta)|^2 r \, d\theta \qquad \forall\, r \in (0, R)$$

$\Rightarrow \forall\, 0 < \delta < R < \frac{1}{2}$,

$$c^2 \log \frac{R}{\delta} \le 2\pi \int_\delta^R \int_{\theta_1(r)}^{\theta_2(r)} |F'(\zeta)|^2 r \, d\theta$$

$$\le 2\pi \int_0^R \int_{\theta_1(r)}^{\theta_2(r)} |F'(\zeta)|^2 r \, d\theta \le 2\pi \, \mathrm{Area}(P)$$

It is a contradiction since $\log \frac{R}{\delta} \to \infty$ as $\delta \to 0$

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