# §4 Conformal Mappings onto Polygons

"Explicit" formula of conformal mapping from $\mathbb{H}$ to polygons.

## 4.1 Some examples

Eg 1. Recall $f(z) = z^\alpha$ is a conformal map from

$\mathbb{H}$ to the sector $\{z : 0 < \arg z < \alpha\pi\}$, $0 < \alpha < 2$

(Eg 2 of section 1, page 210 on the Textbook)

- Note that

$$z^\alpha = f(z) = \int_0^z f'(\zeta)\,d\zeta = \alpha \int_0^z \zeta^{\alpha-1}\,d\zeta$$

denote $\beta = 1-\alpha$, then

$$f(z) = z^\alpha = \alpha \int_0^z \zeta^{-\beta}\,d\zeta \quad \text{with } \alpha+\beta = 1.$$

- The integral can be taken along any path in $\mathbb{H}$.

  Continuity $\Rightarrow$ any path in closure of $\mathbb{H}$,

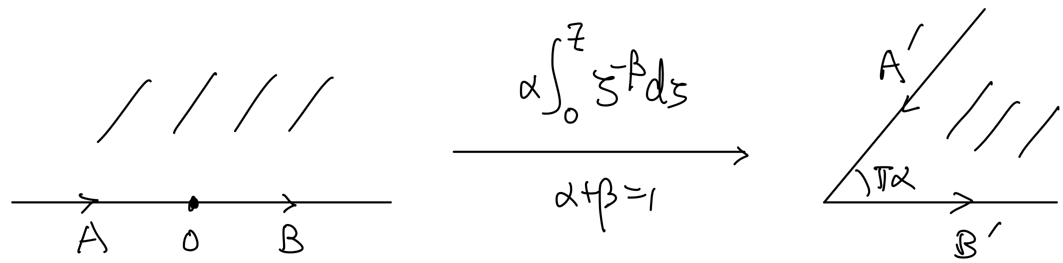
  i.e. including line segments along the $\mathbb{R}$-axis.

- $0 < \alpha < 2 \Rightarrow \beta < 1 \Rightarrow \zeta^{-\beta}$ integrable at $\zeta = 0$.

$$\Rightarrow \begin{cases} f(z) = \int_0^z \zeta^{-\beta}\,d\zeta \text{ defined at } z=0 \text{ and} \\ f(0) = 0 \end{cases}$$

(Originally $z^\alpha = e^{\alpha \log z}$ is not defined for $z = 0$ !)

- Boundary mapping as in the figure

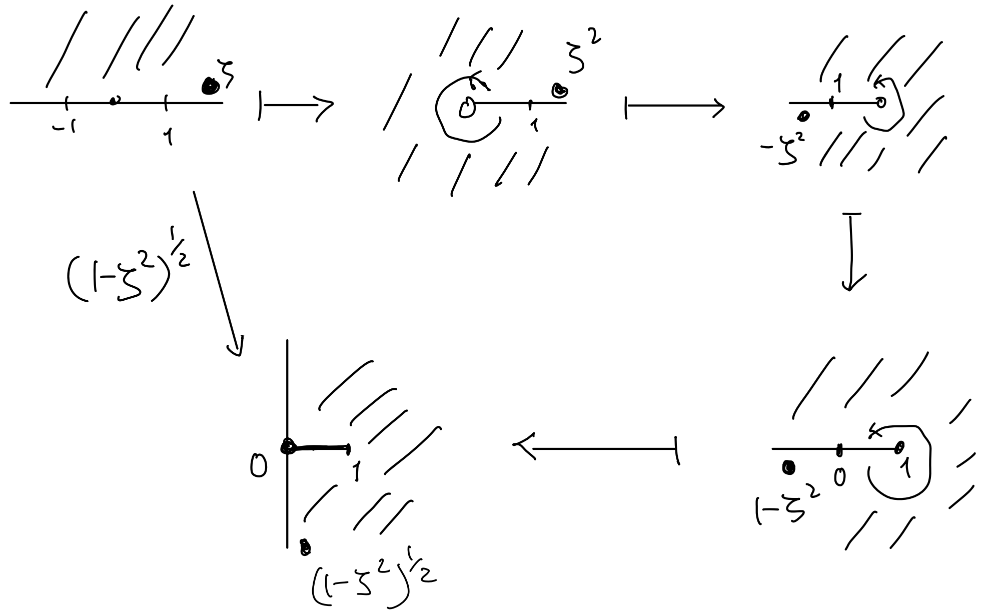



<u>Eg2</u> Consider
$$f : \mathbb{H} \to \mathbb{C}$$
$$z \mapsto \int_0^z \frac{d\zeta}{(1-\zeta^2)^{1/2}}$$
where integral taken along any path in closure$(\mathbb{H})$,

with branch of square root s.t.

(i) $(1-\zeta^2)^{1/2}$ holo in $\mathbb{H}$ ;

(ii) $(1-\zeta^2)^{1/2} > 0$ for $-1 < \zeta < 1$.




- Singular points : $\zeta = \pm 1$ and

$$\int_0^z \frac{d\zeta}{(1-\zeta^2)^{1/2}} = \int_0^z \frac{d\zeta}{(1+\zeta)^{1/2}(1-\zeta)^{1/2}}$$ is integrable !

- For $z = x \in (-1, 1)$,

take path = line segment from 0 to x on $\mathbb{R}$-axis,

$$\int_0^x \frac{d\zeta}{(1-\zeta^2)^{1/2}} = \sin^{-1} x$$

with principal branch $|\sin^{-1} x| < \frac{\pi}{2}$

Taking limits, we see that

$$\int_0^{\pm 1} \frac{d\zeta}{(1-\zeta^2)^{1/2}} = \pm \frac{\pi}{2}$$

- For $\zeta > 1$,

$$\begin{cases} |(1-\zeta^2)^{1/2}| = (\zeta^2-1)^{1/2} \\ \arg (1-\zeta^2)^{1/2} = -\frac{\pi}{2} \end{cases}$$

according to the choose of the branch

(see figure above)

$$\Rightarrow \quad (1-\zeta^2)^{1/2} = -i(\zeta^2-1)^{1/2}$$

$\Rightarrow$ For $x > 1$,

$$f(x) = \int_0^x \frac{d\zeta}{(1-\zeta^2)^{1/2}} = \int_0^1 \frac{d\zeta}{(1-\zeta^2)^{1/2}} + \int_1^x \frac{d\zeta}{(1-\zeta^2)^{1/2}}$$

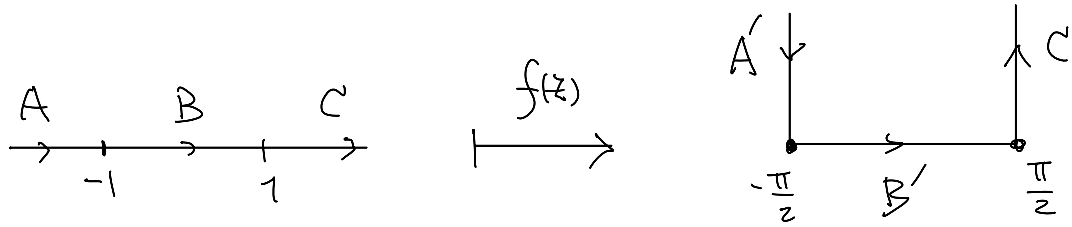
$$= \frac{\pi}{2} + \int_1^x \frac{d\zeta}{-i(\zeta^2-1)^{1/2}}$$

$$= \frac{\pi}{2} + i \int_1^x \frac{d\zeta}{(\zeta^2-1)^{1/2}}$$

$$= \frac{\pi}{2} + i\, \mathrm{ch}^{-1} x \qquad (\mathrm{ch} = \cosh)$$

Similarly for $x<-1$. (Ex!) $\left( f(x) = -\frac{\pi}{2} + i \cosh^{-1}|x| \right)$

Hence $f(z)$ maps the boundary $\mathbb{R}$-line to




- In fact, $f(z) = \sin^{-1} z$ (Ex!)

  (Refer to Eg 8 of section 1)

  $\therefore$ $f$ maps $\mathbb{H}$ conformally onto the half-infinite strip as in the figure.

<u>Eg 3</u> Consider

$$f(z) = \int_0^z \frac{d\zeta}{[(1-\zeta^2)(1-k^2\zeta^2)]^{1/2}}, \quad z \in \mathbb{H}$$

where
- $0<k<1$, $k$ fixed
- the branch of $(1-\zeta^2)^{1/2}$ & $(1-k^2\zeta^2)^{1/2}$ is chosen s.t.

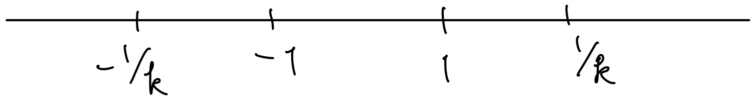
  (i) holo. in $\mathbb{H}$;

  (ii) real & positive for $-1<\zeta<1$.

  and $-\frac{1}{k}<\zeta<\frac{1}{k}$ respectively

- $f(z)$ is an <u>elliptic integral</u> (related to calculating the arc-length of an ellipse).

- There are 4 poles along the $\mathbb{R}$-line




- Clearly integrable as the exponent is $1/2$.

- For $z = x$ with $-1 < x < 1$,

$$f'(x) = \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} > 0 \quad \text{(by the choice of branch)}$$

Together with $f'(-z) = f'(z)$, we have

$$f(\pm 1) = \pm \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

It is tradictionally denote $K = \displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$

Then $f(\pm 1) = \pm K$ and

$f(x)$ increases from $-K$ to $K$

as $x$ increases from $-1$ to $1$.

- For $z = x$ with $1 < x < 1/k$.

Then along the path from $0$ to $x$ on the $\mathbb{R}$-line, we pass through the pole $z = 1$, and the choice of branching of the square root gives

$$\left[(1-\zeta^2)(1-k^2\zeta^2)\right]^{1/2} = -i\sqrt{(\zeta^2-1)(1-k^2\zeta^2)}$$

(as in Eg 2)

Hence

$$f(x) = \int_0^x \frac{d\zeta}{\left[(1-\zeta^2)(1-k^2\zeta^2)\right]^{1/2}}$$

$$= \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \int_1^x \frac{dx}{-i\sqrt{(x^2-1)(1-k^2x^2)}}$$

$$= K + i\int_1^x \frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}}$$

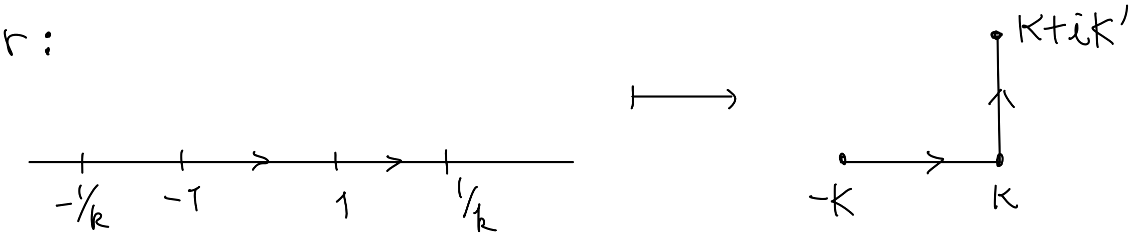
$\therefore$ $f$ maps the segment $(1, 1/k)$ to the vertical segment $K$ to $K + iK'$,

where $$K' = \int_1^{1/k} \frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}}$$

with $f(1) = K$ to $f(1/k) = K + iK'$

as $x$ goes from $1$ to $1/k$.

So far:



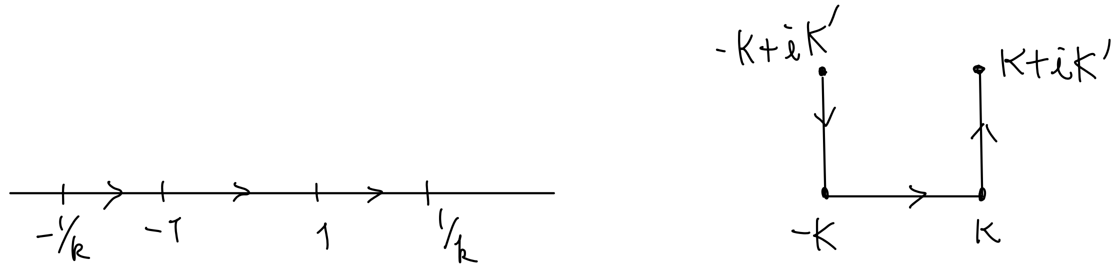
Similarly (Ex!), we have

$f([-\frac{1}{k},-1]) =$ vertical segment with end points $-K$ and $-K+iK'$

s.t. $f(-\frac{1}{k}) = -K+iK'$ to $f(-1) = -K$

as $x$ goes from $-1/k$ to $-1$.

$\therefore$




- For $z = x$ with $x > 1/k$,

we pass thro the pole $1/k$ too, therefore

$$\left[(1-\zeta^2)(1-k^2\zeta^2)\right]^{1/2} = -i\left(-i\sqrt{(x^2-1)(k^2x^2-1)}\right)$$

$$= -\sqrt{(x^2-1)(k^2x^2-1)}$$

$$\therefore \quad f'(x) = -\frac{1}{\sqrt{(x^2-1)(k^2x^2-1)}} < 0$$

And $$f(x) = K + iK' - \int_{1/k}^{x} \frac{dx}{\sqrt{(x^2-1)(k^2x^2-1)}}$$

$\therefore$ $f(x)$ belongs to the horizontal line $y = K'$

Note that $\displaystyle\int_{1/k}^{x} \frac{dx}{\sqrt{(x^2-1)(k^2x^2-1)}} > 0$

and
$$\int_{1/k}^{\infty} \frac{dx}{\sqrt{(x^2-1)(k^2x^2-1)}} = \int_1^0 \frac{-\frac{1}{ku^2}\,du}{\sqrt{\left(\frac{1}{k^2u^2}-1\right)\left(\frac{1}{u^2}-1\right)}} \qquad \left(x=\frac{1}{ku}\right)$$
$$= \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} = K$$

$\therefore$ $f$ maps $(1/k, \infty)$ to the horizontal segment

$(iK', K+iK')$ (in reverse direction)

and $f(\frac{1}{k}) = K+iK'$, $\lim_{x\to+\infty} f(x) = iK'$

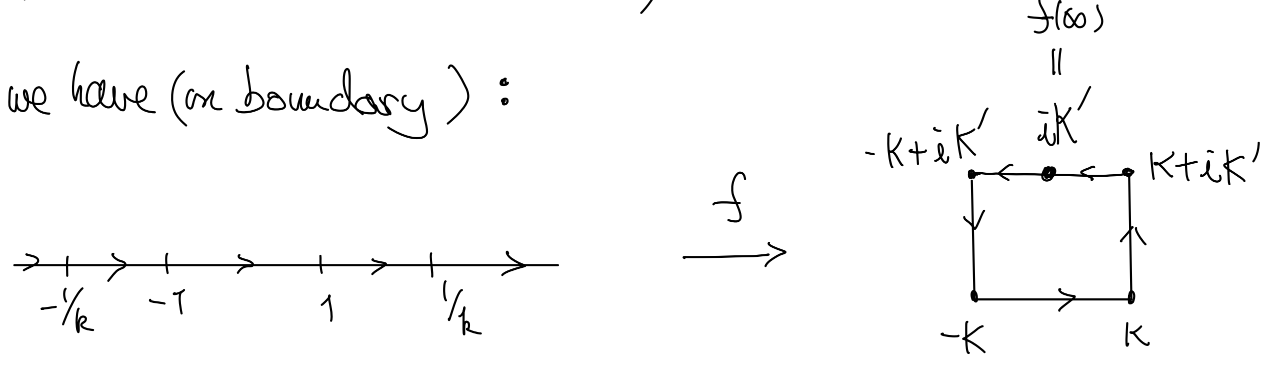
Similarly $f$ maps $(-\infty, -1/k)$ to the horizontal segment

$(-K+iK', iK')$

and $f(-1/k) = -K+iK'$, $\lim_{x\to-\infty} f(x) = iK'$.

$\left(\text{In fact, } \lim_{\substack{z\to\infty \\ z\in \mathbb{H}}} f(z) = iK'\right)$

So we have (on boundary):




(Of course, we haven't shown that $f(\mathbb{H})$ = interior of the rectangle in the figure, nor bijective yet)

## 4.2 The Schwarz-Christoffel Integral

**Def** Schwarz-Christoffel Integral:

$$S(z) = \int_0^z \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\zeta - A_n)^{\beta_n}} \tag{5}$$

where
- $A_1 < \cdots < A_n$ are $n$ distinct points on the real axis;
- $\beta_k < 1$, $\forall\, k = 1, \cdots, n$ such that

$$1 < \sum_{k=1}^{n} \beta_k$$

- branch of $(x - A_k)^{\beta_k}$ is given as in Remark (ii) below

Remarks: (i) In Eg 1, $\beta = 1 - \alpha < 1$

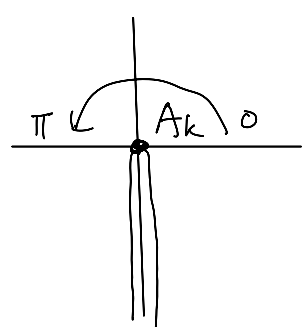
Eg 2, $\beta_1 + \beta_2 = \frac{1}{2} + \frac{1}{2} = 1$

Eg 3, $\beta_1 + \beta_2 + \beta_3 + \beta_4 = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2 > 1$.

In Egs 1 & 2, the image sets are not (bounded) polygons.

(ii) $(z - A_k)^{\beta_k}$ is the branch defined on

$$\mathbb{C} \setminus \{ A_k + iy : y \le 0 \}$$




s.t. $(x - A_k)^{\beta_k} > 0$ for $z = x > A_k$

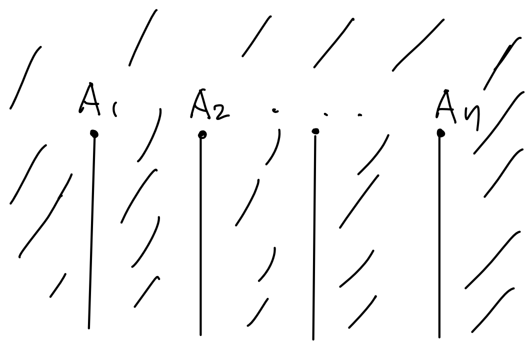
Then $$(x-A_k)^{\beta_k} = \begin{cases} (x-A_k)^{\beta_k}, & \text{if } z=x>A_k \\ |x-A_k|^{\beta_k} e^{i\pi\beta_k}, & \text{if } z=x<A_k \end{cases}$$

(May be a different choose from the examples.)

(iii) Note that

$$\Omega = \mathbb{C} \setminus \bigcup_{k=1}^{n} \{A_k + iy : y \le 0\}$$

is simply-connected,




so $S(z)$ is well-defined and holomaphic in $\Omega$.

Moreover $\beta_k < 1 \Rightarrow \dfrac{1}{|\zeta - A_k|^{\beta_k}}$ is integrable near $A_k$.

(along any path in $\Omega$ to $A_k$)

$\therefore$ $S(z)$ extends continuously to the points $A_k$

with values $\boxed{S(A_k) = a_k}$, $k=1,\cdots,n$

In particular,

$S(z)$ is continuous on $\mathbb{H} \cup \{\text{real-line}\}$

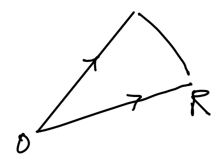
and holo. in $\mathbb{H}$.

(iv) $$\frac{1}{\left|(\zeta-A_1)^{\beta_1}\cdots(\zeta-A_n)^{\beta_n}\right|}=\frac{1}{|\zeta-A_1|^{\beta_1}\cdots|\zeta-A_n|^{\beta_n}}$$

$$\leq \frac{1}{C\,|\zeta|^{\sum_{k=1}^{n}\beta_k}} \quad \text{for } |\zeta| \text{ large}$$

$\therefore \sum_{k=1}^{n}\beta_k > 1 \Rightarrow$ The integral $S(z)$ converges at $\infty$.

$\Rightarrow \boxed{\lim_{r\to\infty} S(re^{i\theta}) = a_\infty}$ exists and independent of $\theta$, $0\leq\theta\leq\pi$.

( Cauchy Thm on  & let $R\to\infty$ )

**Prop 4.1** Suppose $S(z)$ is given by (5) in the above definition and $a_1, \cdots, a_n$ & $a_\infty$ are as in the remarks (iii) & (iv).

(i) If
$$\begin{cases} \bullet\ \sum_{k=1}^{n} \beta_k = 2, \text{ and} \\ \bullet\ \mathcal{P} \text{ denotes the polygon whose vertices are given by } a_1, \cdots, a_n \text{ (in order)} \end{cases}$$

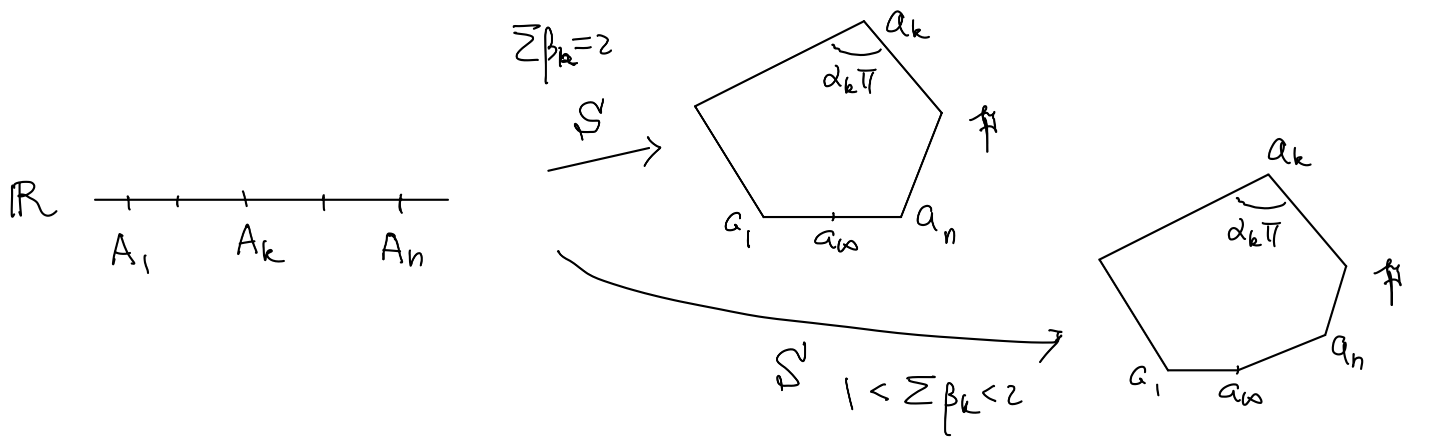
("polygon" = a closed curve consists of finitely many line segments.)

then
$$\begin{cases} \bullet\ a_\infty = S(\infty) \text{ lies on the segment } [a_n, a_1] \\ \bullet\ S(\mathbb{R}) = \mathcal{P} \setminus \{a_\infty\} \\ \bullet\ \text{(Interior) angle at } a_k = \alpha_k \pi, \quad \alpha_k = 1 - \beta_k. \end{cases}$$

(ii) If $1 < \sum_{k=1}^{n} \beta_k < 2$, the similar conclusion holds with

$$\begin{cases} \bullet\ \mathcal{P} \text{ replaced by the polygon of } n+1 \text{ sides with vertices } a_1, a_2, \cdots a_n, a_\infty \text{ (in order), and} \\ \bullet\ \text{(Interior) angle at } a_\infty = \alpha_\infty \pi, \end{cases}$$

$$\alpha_\infty = 1 - \beta_\infty \quad \& \quad \beta_\infty = 2 - \sum_{k=1}^{n} \beta_k.$$



Pf Case (i) $\sum_{k=1}^{n} \beta_k = 2$

If $A_k < x < A_{k+1}$, $k = 1, \cdots, n-1$.

Then $$S'(x) = \frac{1}{\left[(x-A_1)^{\beta_1} \cdots (x-A_k)^{\beta_k}\right]\left[(x-A_{k+1})^{\beta_{k+1}} \cdots (x-A_n)^{\beta_n}\right]}$$

By the choice of branch of each $x - A_j$ in Remark (ii),

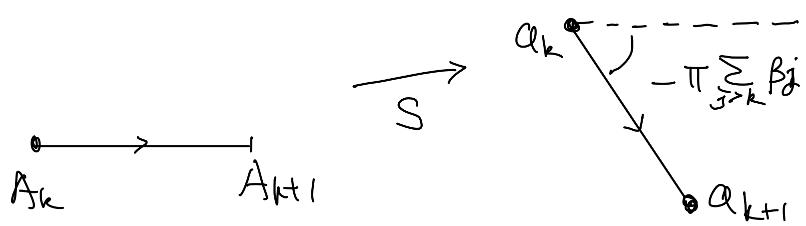
$$\arg (x-A_j)^{\beta_j} = \begin{cases} 0 & \text{for } j \le k \\ \pi\beta_j & \text{for } j > k \end{cases}$$

$\therefore$ $\arg S'(x) = -\pi \sum_{j>k} \beta_j$ which is a constant for $x \in (A_k, A_{k+1})$.

$\Rightarrow$ $S[A_k, A_{k+1}]$ is a straight line segment that makes an angle of $-\pi \sum_{j>k} \beta_j$ with the x-axis.

Notice that $S(x) = S(A_k) + \int_{A_k}^{x} S'(y)\,dy \quad \forall x \in (A_k, A_{k+1})$.

$S(x)$ varies from end point $a_k = S(A_k)$ to end point $a_{k+1} = S(A_{k+1})$ as $x$ varies from $A_k$ to $A_{k+1}$.




Similarly
$$\arg S'(x) = \begin{cases} 0 & \text{if } x > A_n \quad (\text{i.e. } S'(x) > 0) \\ -\pi \sum_{k=1}^{n} \beta_k = -2\pi, & \text{if } x < A_1 \end{cases}$$

And • $S(x)$ varies from $a_n = S(A_n)$ to $a_\infty = S(A_\infty)$

as $x$ varies from $A_n$ to $\infty$.

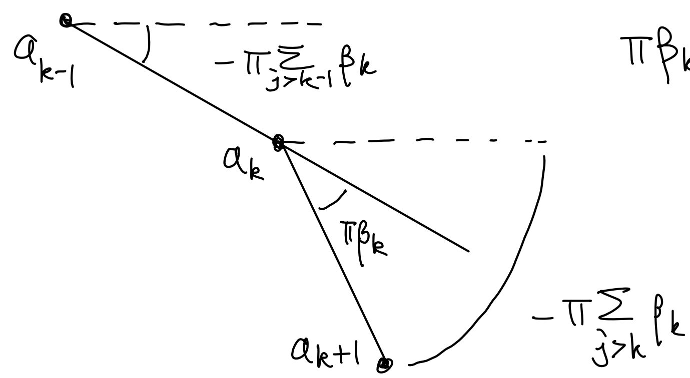
• $S(x)$ varies from $a_\infty$ to $a_1 = S(A_1)$

as $x$ varies from $-\infty$ to $A_1$

This shows that $a_\infty \in [a_1, a_n]$ (angles with x-axis $= 0$ & $-2\pi$)

This proves $S(\mathbb{R}) = P \setminus \{a_\infty\}$.

Note that

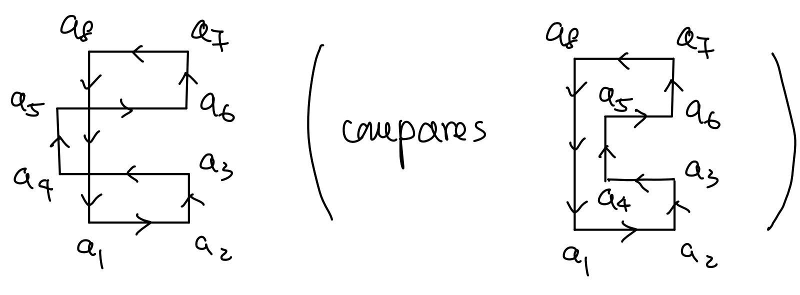



$$\pi\beta_k = \Big(-\pi \sum_{j>k} \beta_k\Big) - \Big(-\pi \sum_{j>k-1} \beta_k\Big)$$

$\therefore$ Interior angle at $a_k = \pi - (\pi\beta_k) = \alpha_k \pi$.

Case (ii) $1 < \sum_{k=1}^{n} \beta_k < 2$ is similar (Ex!) ##

Notes: (i) For an arbitrary choice of $n$; $A_1, \cdots, A_n$; $\beta_1, \cdots, \beta_n$, the "polygon" $P$ in Prop 4.1 may not be simple. The following could happen:



(ii) Even $\mathfrak{P} = \partial P$, $P$ simply-connected region, Prop 4.1 hasn't shown that $S : \mathbb{H} \to P$ is conformal. (See subsection 4.4 below)