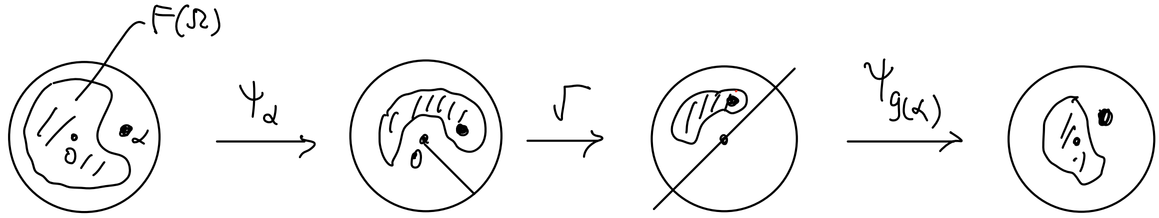
Step 4: The injective holo. map F found in Step 3 is conformal $F: \Omega \to \mathbb{D}$.

Pf: It remains to show: $F(\Omega) = \mathbb{D}$ (i.e. F is onto)

Suppose on the contrary that $\mathbb{D} \setminus F(\Omega) \neq \emptyset$.

i.e. $\exists\, \alpha \in \mathbb{D} \setminus \{0\}$ s.t. $F(z) \neq \alpha$, $\forall z \in \Omega$.

Then $$\psi_\alpha \circ F(z) = \frac{\alpha - F(z)}{1 - \bar{\alpha} F(z)} \neq 0 \quad \forall z \in \Omega$$



Since Ω is simply-connected, $U = \psi_\alpha \circ F(\Omega)$ is also simply-connected as ψ_α & F are conformal to their respective images.

Hence $g: U \to \mathbb{C} : w \mapsto w^{\frac{1}{2}} = e^{\frac{1}{2}\log w}$

can be defined.

Consider holo. $f = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ F$.

Then $|\psi_\alpha \circ F| < 1 \Rightarrow |g \circ \psi_\alpha \circ F| = |\psi_\alpha \circ F|^{\frac{1}{2}} < 1$

$\therefore \quad f : \Omega \to \mathbb{D}$

Clearly, f is injective as square root g, ψ_α, $\psi_{g(\alpha)}$ and F are injective.

$$f(0) = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ F(0)$$

$$= \psi_{g(\alpha)} \circ g \circ \psi_\alpha(0) = \psi_{g(\alpha)}(g(\alpha)) = 0$$

$$\left(\psi_{g(\alpha)}(z) = \frac{g(\alpha) - z}{1 - \overline{g(\alpha)} z} \right)$$

$\therefore$ $f \in \mathcal{F}$.

Let $h(w) = w^2$, then

$$f = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ F \qquad (\psi_\alpha \circ \psi_\alpha = Id)$$

$$\Rightarrow \quad \psi_{g(\alpha)} \circ f = g(\psi_\alpha \circ F) \qquad (g: \text{square root})$$

$$\Rightarrow \quad \psi_\alpha \circ F = h(\psi_{g(\alpha)} \circ f) \qquad (h: \text{square})$$

$$\Rightarrow \quad F = (\psi_\alpha \circ h \circ \psi_{g(\alpha)}) \circ f$$

$$= \Phi \circ f.$$

Note that • $\Phi : \mathbb{D} \to \mathbb{D}$ holo. as ψ_α, $\psi_{g(\alpha)}$ & $h : \mathbb{D} \to \mathbb{D}$ holo.

• $\Phi(0) = \psi_\alpha \circ h \circ \psi_{g(\alpha)}(0)$

$$= \psi_\alpha \circ h(g(\alpha)) = \psi_\alpha(\alpha) = 0$$

• Φ is not a rotation.

In fact, if Φ is a rotation, then

$$e^{i\theta} g(\alpha) = \Phi(g(\alpha)) = \psi_\alpha \circ h(0) = \psi_\alpha(0) = \alpha$$

$\Rightarrow \ |\alpha|^{\frac{1}{2}} = |\alpha|$

$\Rightarrow \ |\alpha| = 1$ or $\alpha = 0$

which is a contradiction as $\alpha \in \mathbb{D}$ and

$\alpha \notin F(\Omega)$ since $0 = F(0) \in F(\Omega)$.

Then Schwarz Lemma $\Rightarrow$

$$|\Phi'(0)| < 1$$

Hence

$$\sup_{\zeta \in \mathcal{F}} |\zeta'(0)| = s = |F'(0)| = |\Phi'(0)|\,|f'(0)| < |f'(0)|$$

which is a contradiction.

Hence $F(\Omega) = \mathbb{D}$.

Final Step: Choose $\theta \in \mathbb{R}$ suitably to conclude

$$\begin{cases} e^{i\theta} F(z) : \Omega \to \mathbb{D} \text{ is conformal,} \\ e^{i\theta} F(0) = 0 \\ (e^{i\theta} F)'(0) > 0 \end{cases}$$

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