Explicit description of Aut($\mathbb{H}$)

> **Thm 2.4**: $f \in \text{Aut}(\mathbb{H}) \Leftrightarrow$
>
> $$f(z) = \frac{az+b}{cz+d} \quad \text{for some } a,b,c,d \in \mathbb{R}$$
> $$\& \quad ad-bc = 1. \ (>0 \text{ suff.})$$

Remarks: (i) $a,b,c,d \in \mathbb{R}$, not just $\mathbb{C}$.

(ii) any $f \in \text{Aut}(\mathbb{H})$ is a fractional linear transformation

(iii) any $f \in \text{Aut}(\mathbb{D})$ is a fractional linear transformation

Pf of Thm 2.4

($\Leftarrow$) $ad-bc=1 \Rightarrow (a,b),(c,d)$ are linearly independent

(in particular c,d can't be 0 simultaneously)

$\therefore \ f(z) = \dfrac{az+b}{cz+d}$ is well-defined (and non-constant)

$c,d \in \mathbb{R} \Rightarrow f$ is holo in $\mathbb{H}$.

Now $$f(x+iy) = \frac{a(x+iy)+b}{c(x+iy)+d} = \frac{(ax+b)+iay}{(cx+d)+icy}$$

$$= \frac{[(ax+b)+iay][(cx+d)-icy]}{(cx+d)^2+c^2y^2}$$

$$\Rightarrow \ \text{Im} f(z) = \frac{ay(cx+d)-cy(ax+b)}{(cx+d)^2+c^2y^2} = \frac{(ad-bc)y}{|cz+d|^2}$$

$$= \frac{y}{|cz+d|^2} > 0 \quad \forall y > 0$$

$\therefore \; f : \mathbb{H} \to \mathbb{H}$.

Observe that $g(z) = \dfrac{dz-b}{-cz+a}$ has the same form

with coefficients satisfying $d \cdot a - (-b)(-c) = 1$.

$\therefore$ g is well-defined, holo in $\mathbb{H}$ and

$g : \mathbb{H} \to \mathbb{H}$.

Straight forward calculation:

$$f \circ g(z) = \frac{a\left(\frac{dz-b}{-cz+a}\right)+b}{c\left(\frac{dz-b}{-cz+a}\right)+d} = \frac{a(dz-b)+b(-cz+a)}{c(dz-b)+d(-cz+a)}$$

$$= \frac{(ad-bc)z}{(ad-bc)} = z$$

Similarly $g \circ f(z) = z$.

$\therefore$ $g = f^{-1}$ and hence $f \in \mathrm{Aut}(\mathbb{H})$.

($\Rightarrow$) If $f \in \mathrm{Aut}(\mathbb{H})$, then $\beta \overset{\text{denote}}{=} f^{-1}(i) \in \mathbb{H}$.

If $\beta = u + iv$, $u, v \in \mathbb{R}$, $v > 0$.

Then $\psi(z) = \dfrac{z-u}{v} = \dfrac{\frac{1}{\sqrt{v}} z + \left(-\frac{u}{\sqrt{v}}\right)}{0 \cdot z + \sqrt{v}} \in \mathrm{Aut}(\mathbb{H})$

as $\frac{1}{\sqrt{v}} \cdot \sqrt{v} - \left(-\frac{u}{\sqrt{v}}\right) \cdot 0 = 1$.

And $\psi(\beta) = \dfrac{(u+iv)-u}{v} = i$ & hence $\psi^{-1}(i) = \beta$

Consider $g = f \circ \psi^{-1} \in \mathrm{Aut}(\mathbb{H})$.

Then $g(i) = f \circ \psi^{-1}(i) = f(\beta) = i$.

$\Rightarrow$ $\Gamma^{-1}(g) = F \circ g \circ F^{-1} \in \mathrm{Aut}(\mathbb{D})$, where $F(z) = \dfrac{i-z}{i+z}$,

satisfies $\Gamma^{-1}(g)(0) = F \circ g \circ F^{-1}(0) = F \circ g(i) = F(i) = 0$

Schwarz Lemma $\Rightarrow$ $\Gamma^{-1}(g)(z) = e^{i2\theta} z$ for some $\theta \in \mathbb{R}$

$$\Rightarrow \quad g(z) = F^{-1} \circ \Gamma^{-1}(g) \circ F(z) = F^{-1}\left(e^{i2\theta} F(z)\right)$$

$$= i \, \frac{1 - e^{i2\theta}\left(\frac{i-z}{i+z}\right)}{1 + e^{i2\theta}\left(\frac{i-z}{i+z}\right)}$$

$$= i \, \frac{(1+e^{i2\theta})z + i(1-e^{i2\theta})}{(1-e^{i2\theta})z + i(1+e^{i2\theta})}$$

$$= i \, \frac{(e^{i\theta}+e^{-i\theta})z - i(e^{i\theta}-e^{-i\theta})}{-(e^{i\theta}-e^{-i\theta})z + i(e^{i\theta}+e^{-i\theta})}$$

$$\Rightarrow \quad f \circ \psi^{-1}(z) = \frac{\cos\theta \cdot z + \sin\theta}{-\sin\theta \cdot z + \cos\theta}$$

$$\therefore \quad f(z) = \frac{\cos\theta \cdot \left(\frac{z-u}{v}\right) + \sin\theta}{-\sin\theta \cdot \left(\frac{z-u}{v}\right) + \cos\theta}$$

$$= \frac{\frac{\cos\theta}{\sqrt{v}} \cdot z + \frac{(-u\cos\theta + v\sin\theta)}{\sqrt{v}}}{-\frac{\sin\theta}{\sqrt{v}} \cdot z + \frac{(u\sin\theta + v\cos\theta)}{\sqrt{v}}}$$

Clearly coefficients are real and

$$\frac{\cos\theta}{\sqrt{v}}\frac{(u\sin\theta+v\cos\theta)}{\sqrt{v}} - \frac{(-\sin\theta)}{\sqrt{v}}\frac{(-u\cos\theta+v\sin\theta)}{\sqrt{v}}$$

$$= \cos^2\theta + \sin^2\theta = 1$$

$\therefore$ f is of the required form. ※

<u>Remark</u>: The proof in the Textbook uses the following relationship between fractional linear transformations and 2×2 matrixes.

$$f_M(z) = \frac{az+b}{cz+d} \longleftrightarrow M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that

(i) $f_I = \mathrm{Id}$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(ii) $f_{M_1} \circ f_{M_2} = f_{M_1 M_2}$

Pf: $$\frac{a_1\left(\frac{a_2z+b_2}{c_2z+d_2}\right)+b_1}{c_1\left(\frac{a_2z+b_2}{c_2z+d_2}\right)+d_1} = \frac{a_1(a_2z+b_2)+b_1(c_2z+d_2)}{c_1(a_2z+b_2)+d_1(c_2z+d_2)}$$

$$= \frac{(a_1a_2+b_1c_2)z+(a_1b_2+b_1d_2)}{(c_1a_2+d_1c_2)z+(c_1b_2+d_1d_2)}$$

(provided $(c_1,d_1)\neq 0$ & $(c_2,d_2)\neq 0$)

(iii) By (i) & (ii), $(f_M)^{-1}$ exists $\Leftrightarrow$ M^{-1} exists

and $(f_M)^{-1} = f_{(M^{-1})}$

(ii) However, $f_{(-M)} = f_M$ ($\Rightarrow$ not 1-1 correspondence)

(In fact $f_{(kM)} = f_M$, $\forall k \in \mathbb{C}\setminus\{0\}$)

For the purpose of proving Thm 2.4, the Textbook considered

$$SL_2(\mathbb{R}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \ \& \ \det(M) = ad - bc = 1 \right\}$$

(real) <u>special linear group</u> (of degree 2)

and Thm 2.4 can be written as

$$\underset{\left(\substack{\cong \\ \mathrm{Aut}(\mathbb{D})}\right)}{\mathrm{Aut}(\mathbb{H})} \overset{\text{group iso}}{\cong} SL_2(\mathbb{R}) / \pm I \overset{\text{def}}{=\!=} PSL_2(\mathbb{R}) \quad \left(\begin{array}{l}\text{(real) projective} \\ \text{special linear group} \\ \text{(of degree 2)}\end{array}\right).$$

§3 The Riemann Mapping Theorem

3.1 Necessary Conditions and Statement of the Theorem

The Problem: determine conditions on an (nonempty) open set Ω that guarantee the existence of conformal map $F: \Omega \to \mathbb{D}$.

(Then for Ω satisfying these conditions, Dirichlet problem in Ω is solvable.)

Necessary conditions

(1) If $F: \Omega \to \mathbb{D}$ conformal, then

$$\sup_{z \in \Omega} |F(z)| = 1$$

Therefore $\Omega \neq \mathbb{C}$, otherwise Liouville's Thm

$\Rightarrow$ $F(z) \equiv$ const. which cannot be conformal.

For convenience, let we call a non-empty set Ω proper if $\Omega \neq \mathbb{C}$.

(2) If $F: \Omega \to \mathbb{D}$ conformal, then $F: \Omega \to \mathbb{D}$ is a homeomorphism and hence Ω and $\mathbb{D}$ are topological equivalent. In particular,

Ω must be a simply-connected region. (open and connected in $\mathbb{C}$)

Thm 3.1 (**Riemann Mapping Theorem**)

Suppose **region** Ω is **proper** and **simply-connected**.
Then $\forall\, z_0 \in \Omega$, $\exists$ a **unique** **conformal** map

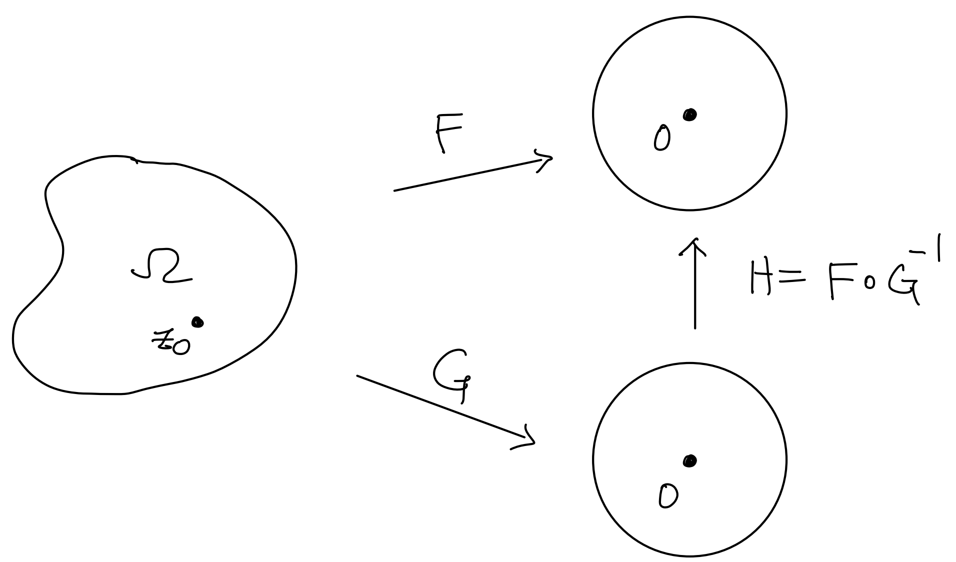
$$F : \Omega \to \mathbb{D} \quad \text{such that } F(z_0) = 0 \text{ and } F'(z_0) > 0.$$

This means $F'(z_0) \in \mathbb{R}$ and $F'(z_0) > 0$.

Cor 3.2 Any two **proper** **simply connected** **regions** in $\mathbb{C}$ are conformally equivalent.

Remark: Hence simply connected regions in $\mathbb{C}$ fall into only 2 conformal equivalent classes: $\mathbb{C}$ or $\mathbb{D}$.

Proof of uniqueness of Thm 3.1.



Suppose that $F: \Omega \to \mathbb{D}$, $G: \Omega \to \mathbb{D}$ are conformal

and satisfying
$$\begin{cases} F(z_0) = G(z_0) = 0 \\ F'(z_0) > 0, \quad G'(z_0) > 0. \end{cases}$$

Then $H = F \circ G^{-1} : \mathbb{D} \to \mathbb{D}$ conformal,

and $H(0) = F \circ G^{-1}(0) = F(z_0) = 0$

$\therefore \quad H \in \mathrm{Aut}_0(\mathbb{D})$.

By Schwarz Lemma (more precisely Cor 2.3),

$$H(z) = e^{i\theta} z \qquad \text{for some } \theta \in \mathbb{R}.$$

$$\Rightarrow \quad e^{i\theta} = H'(0) = F'(G^{-1}(0)) \frac{1}{G'(G^{-1}(0))} = \frac{F'(z_0)}{G'(z_0)} > 0$$

real and positive

$\therefore \quad e^{i\theta} = 1$

And hence $F \circ G^{-1}(z) = z \Leftrightarrow F \equiv G$. ※

Existense part is much harder and will be handled in the next two subsections.

3.2 Montel's Theorem

Def: Let $\Omega \subset \mathbb{C}$ be open. A family $\mathcal{F}$ of holomorphic functions on Ω is said to be normal if every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset of Ω

($\mathcal{F}$ is called precompact if one can make the convergence as a convergence of a metric space (Ω, d). See MATH3060.)

Def: Let $\Omega \subset \mathbb{C}$ be open. A family $\mathcal{F}$ of holomorphic functions on Ω is said to be

(1) uniformly bounded on compact subsets of Ω

if $\forall$ compact set $K \subset \Omega$, $\exists B > 0$ such that

$$|f(z)| \leq B, \ \forall z \in K \text{ and } f \in \mathcal{F}.$$

(2) equicontinuous on a compact set K

if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

whenever $z, w \in K$ with $|z - w| < \delta$,

then $|f(z) - f(w)| < \varepsilon$, $\forall f \in \mathcal{F}$.

(Ex: review MATH3060 on the related properties)

In metric space setting of family of continuous functions, the properties (1) and (2) are independent. However, for family of holomorphic functions, (1) $\Rightarrow$ (2), thanks to the Cauchy Integral Formula:

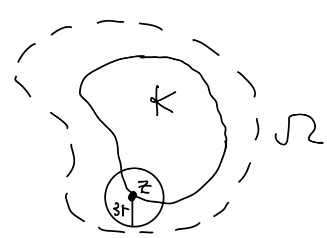
> **Thm 3.3** Suppose $\mathcal{F}$ is a family of holomorphic functions on Ω that is uniformly bounded on compact subsets of Ω.
>
> Then (i) $\mathcal{F}$ is equicontinuous on every compact subset of Ω.
>
> (ii) $\mathcal{F}$ is a normal family.

Pf of (i)

Let $K \subset \Omega$ be compact.

Then $\exists\, r > 0$ such that $\forall z \in K$, $D_{3r}(z) \subset \Omega$

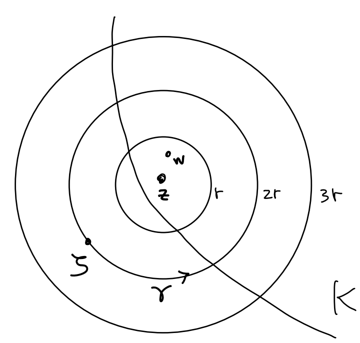
$(0 < r < \frac{1}{3}\operatorname{dist}(K, \partial\Omega))$



If $z, w \in K$ and $|z - w| < r$.

Let $\gamma = \partial D_{2r}(w)$

Then Cauchy's integral formula



$\Rightarrow$ $f(z) - f(w) =$

$$\frac{1}{2\pi i}\int_{\gamma} f(\zeta)\left[\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right] d\zeta$$

$$\Rightarrow \quad |f(z)-f(w)| \le \frac{1}{2\pi}\int_\gamma |f(\zeta)| \left| \frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right| |d\zeta|$$

$$= \frac{1}{2\pi}\int_\gamma |f(\zeta)| \frac{|z-w|}{|\zeta - z||\zeta - w|} |d\zeta|$$

$$\le \frac{1}{2\pi}\int_\gamma |f(\zeta)| \frac{|z-w|}{r^2} |d\zeta|$$

By assumption, $\exists\, B>0$ s.t. $|f(z)| \le B$, $\forall\, z\in\Omega$ & $f\in\mathcal{F}$,

we have $|f(z)-f(w)| \le \dfrac{B|z-w|}{r^2}\cdot\dfrac{1}{2\pi}\cdot 2\pi(2r)$

$$= \frac{2B}{r}|z-w|$$

$\forall\, z,w\in K$, $|z-w|<r$ & $\forall f\in\mathcal{F}$.

This implies $\mathcal{F}$ is equicontinuous ※

To prove (ii), we need the following

Lemma 3.4 Any open set Ω $(\subset \mathbb{C})$ has a compact exhaustion

Recall:

A compact exhaustion (simple called exhaustion in the Textbook) of Ω is a sequence $\{K_\ell\}_{\ell=1}^{\infty}$ of compact subsets of Ω such that

(i) $K_\ell \subset \text{int}(K_{\ell+1})$ $\forall\ \ell = 1, 2, 3, \cdots$

(ii) $\forall$ compact subset K of Ω, $\exists\ K_\ell$ such that

$$K \subset K_\ell.$$

In particular, $\Omega = \bigcup_{\ell=1}^{\infty} K_\ell$.

<u>Pf of Lemma 3.4</u>:

If Ω is bounded, $K_\ell = \{ z \in \Omega : \text{dist}(z, \partial\Omega) \geq \frac{1}{\ell} \}$

is the required compact exhaustion.

If Ω is unbounded,

$$K_\ell = \{ z \in \Omega : \text{dist}(z, \partial\Omega) \geq \tfrac{1}{\ell} \text{ and } |z| \leq \ell \}$$

is the required compact exhaustion.

(Ex: give the details) ※

<u>Pf of (ii) (of Thm 3.3)</u>

Let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ be a sequence.

Let $K \subset \Omega$ be compact.

Then by (i), $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous on K.

Arzela-Ascoli Theorem (on the metric space $(C(K, d_\infty))$)

(review MATH3060)

$\Rightarrow$ $\exists$ subsequence of $\{f_n\}$ converges uniformly on K.

Let $\{K_\ell\}_{\ell=1}^{\infty}$ be a compact exhaustion of Ω.

Then $\{f_n\}$ has a convergent subsequence $\{g_{n,1}\}$ on K_1 (in uniform metric)

Applying the same argument,

$\{g_{n,1}\}$ has a convergent subseq $\{g_{n,2}\}$ on $K_2 \supset K_1$ (in uniform metric)

And so on, we have subseq. $\{g_{n,\ell}\}$ of $\{f_n\}$ satisfying

(i) $\{g_{n,\ell}\}$ converges uniformly on $K_\ell \supset \cdots \supset K_1$

(ii) $\{g_{n,\ell+1}\}$ is a subseq. of $\{g_{n,\ell}\}$.

Then the seq. $\{g_{n,n}\}$ is a subsequence of $\{f_n\}$ that converges uniformly on K_ℓ, $\forall \ell = 1, 2, \cdots$

Since $\{K_\ell\}$ is a compact exhaustion of Ω, $\{g_{n,n}\}$ converges uniformly in any compact $K \subset \Omega$ (as $K \subset K_\ell$ for some ℓ).

This proves $\mathcal{F}$ is normal. #

Prop 3.5 Let
- $\Omega \subset \mathbb{C}$ be a region, &
- $\{f_n\}$, f be holo functions on Ω such that
- $f_n \to f$ uniformly on every compact subset of Ω

If f_n are injective, then

f is either injective or constant.

Pf: Suppose that f is not injective.

Then $\exists\ z_1, z_2 \in \Omega$ such that

$$z_1 \neq z_2 \text{ but } f(z_1) = f(z_2).$$

Define $g_n(z) = f_n(z) - f_n(z_1)$.

Then
$$\begin{cases} g_n(z_1) = 0 \ \& \\ g_n(z) \neq 0, \ \forall z \in \Omega \setminus \{z_1\}, \end{cases}$$ since f_n injectives

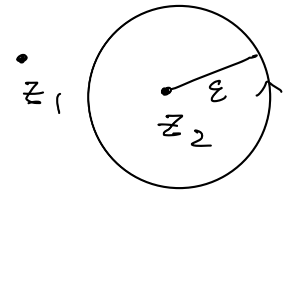
As $f_n \to f$ uniformly on cpt subset,

$$g_n \to g = f - f(z_1) \text{ uniformly on cpt. subset.}$$

If $g \not\equiv 0$, then z_2 is an isolated zero of g.

$$(g(z_2) = f(z_2) - f(z_1) = 0)$$

$$\Rightarrow \ 1 \le \frac{1}{2\pi i} \int_{|\zeta - z_2| = \varepsilon} \frac{g'(\zeta)}{g(\zeta)} d\zeta$$



along a small circle $|\zeta - z_2| = \varepsilon$ around z_2

s.t. $g(\zeta) \neq 0$, $\forall\ |\zeta - z_2| \le \varepsilon$.

Then $\frac{1}{g_n} \to \frac{1}{g}$ uniformly on $|\zeta - z_2| = \varepsilon$

and hence $\displaystyle \frac{1}{2\pi i} \int_{|\zeta - z_2| = \varepsilon} \frac{g_n'(\zeta)}{g_n(\zeta)}\, d\zeta \to \frac{1}{2\pi i} \int_{|\zeta - z_2| = \varepsilon} \frac{g'(\zeta)}{g(\zeta)}\, d\zeta \geq 1$

This is a contradiction as g_n has no zero in $|\zeta - z_2| \leq \varepsilon$

$$\Rightarrow \quad \frac{1}{2\pi i} \int_{|\zeta - z_2| = \varepsilon} \frac{g_n'(\zeta)}{g_n(\zeta)}\, d\zeta = 0, \qquad \forall n,$$

$\therefore \quad g \equiv 0 \quad \Rightarrow \quad f(z) = f(z_1)$ a constant $\forall z \in \Omega$

$\divideontimes$

<u>Remark</u>: The argument in the proof of Prop 3.5 gives the following

<u>Hurwitz Theorem</u>:

If f_n & f analytic in Ω, $f_n(z) \neq 0$, $\forall z \in \Omega$, and f_n converges uniformly to f on every compact set of Ω,

then either (i) $f(z) \equiv 0$, or

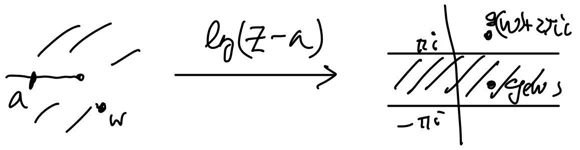
(ii) $f(z) \neq 0$, $\forall z \in \Omega$.

And clearly Hurwitz Thm $\Rightarrow$ Prop 3.5.

3.3 Proof of the Riemann Mapping Theorem

> **Step 1** For a proper and simply-connected region Ω,
>
> and $z_0 \in \Omega$, $\exists$ conformal
>
> $f : \Omega \to f(\Omega) \subset \mathbb{D}$ s.t. $f(z_0) = 0$ & $f'(z_0) > 0$

Pf: Ω is proper $\Rightarrow \exists\, a \in \mathbb{C} \setminus \Omega$.



Then Ω is simply-connected $\Rightarrow$

$g(z) = \log(z-a)$ is well-defined in Ω.

Clearly (i) g is injective

(ii) $\forall\, w \in \Omega$, $g(z) \neq g(w) + 2\pi i$, $\forall\, z \in \Omega$

(by taking exponential)

Claim: $\forall\, w \in \Omega$, $\exists\, r > 0$ s.t. $\overline{D_r}(g(w) + 2\pi i) \cap g(\Omega) = \emptyset$.

Pf: Suppose not, then $\exists\, z_n \in \Omega$ such that

$$g(z_n) \to g(w) + 2\pi i.$$

Taking exponential, $z_n \to w$.

And hence $g(z_n) \to g(w)$ which is a contradiction.

Then $h(z) = \dfrac{r}{g(z) - (g(w) + 2\pi i)}$ is holo. injective and $$|h(z)| = \frac{r}{|g(z)-(g(w)+2\pi i)|} < \frac{r}{r} = 1$$

$\therefore \quad h: \Omega \to h(\Omega) \subset \mathbb{D}$ conformal

Finally, $$f(z) = e^{i\theta}\, \frac{h(z_0)-h(z)}{1-\overline{h(z_0)}\,h(z)} = e^{i\theta}\, \psi_{h(z_0)} \circ h(z)$$

(where ψ_α as in subsection 2.1 & $\theta \in \mathbb{R}$ to be chosen)

is holo. injective, $f(\Omega) \subset \mathbb{D}$, and $f(z_0) = 0$.

Furthermore $$f'(z_0) = e^{i\theta}\, \psi'_{h(z_0)}(h(z_0))\, h'(z_0).$$

Hence if $$\theta = -\arg\left(\psi'_{h(z_0)}(h(z_0))\,h'(z_0)\right),$$

then $$f'(z_0) > 0. \quad ※$$

Step 2: The proof can be reduced to the case that Ω is a simply-connected region in $\mathbb{D}$ with $z_0 = 0 \in \Omega$.

Pf If Riemann Mapping Thm holds in the case as in Step 2,

then $\exists$ conformal $f_1: f(\Omega) \to \mathbb{D}$ conformal,

$$f_1(0) = 0 \;\&\; f_1'(0) > 0,$$

where f is given in Step 1.

Then $F = f_1 \circ f : \Omega \to \mathbb{D}$ is conformal and $\quad F(z_0) = f_1(f(z_0)) = f_1(0) = 0.$

$$F'(z_0) = f_1'(0) f'(z_0) > 0 \quad \divideontimes$$

$$\boxed{\begin{array}{l} \underline{\text{Step 3}} : \text{For simply-connected region } \Omega \subset \mathbb{D} \text{ containing } 0, \\ \quad \exists\, F \in \mathcal{F} = \{ f : \Omega \to \mathbb{D} : \text{holo., injective \& } f(0) = 0 \} \\ \text{s.t.} \qquad |F'(0)| = \sup\limits_{f \in \mathcal{F}} |f'(0)| \end{array}}$$

$\underline{\text{Pf}}$: Clearly $f : \Omega^{\subset \mathbb{D}} \to \mathbb{D} : z \mapsto z \in \mathcal{F}$

$\therefore \quad \mathcal{F} \neq \emptyset.$

This also implies $S = \sup\limits_{f \in \mathcal{F}} |f'(0)| \geq 1.$

On the other hand, by Cauchy inequality (Cor 4.3 in Ch 2)

$$S = \sup_{f \in \mathcal{F}} |f'(0)| < \infty \qquad \left(\text{since } f \in \mathcal{F} \Rightarrow |f| \leq 1\right)$$

Hence $\exists\, f_n \in \mathcal{F}$ such that

$$|f_n'(0)| \to S \quad \text{as } n \to \infty.$$

By Montel's Theorem (Thm 3.3), $\mathcal{F}$ is normal.

(as $\mathcal{F}$ is uniformly bounded)

$\Rightarrow \exists$ subseq (let call it f_n again) converges uniformly on every compact subset to a holo f on Ω.

Then $f(0) = 0$ and $|f'(0)| = s$

$$s \geq 1 \Rightarrow f \not\equiv \text{constant}.$$

Hence Prop 3.5 $\Rightarrow$ f is injective, as f_n are injective.

$\therefore f \in \mathcal{F}$. This proves step 3. ※