(1) Let F holomorphic on $\{z : \operatorname{Re} z > 0\}$ and continuous up to the boundary (the imaginary axis). Suppose

$$|F(iy)| \le 1, \quad \forall y \in \mathbb{R}$$

and $\quad |F(z)| \le C_1 e^{C_2 |z|^\gamma}$ for some $\gamma < 1$, and, $C_1, C_2 > 0$.

Prove that $|F(z)| \le 1 \quad \forall z \in \{z : \operatorname{Re} z \ge 0\}$.

Soln: Take α s.t. $\gamma < \alpha < 1$ and consider $\forall \varepsilon > 0$

$$G_\varepsilon(z) = F(z) e^{-\varepsilon z^\alpha} \quad \text{using principle branch of log.}$$

Then $|G_\varepsilon(z)| = |F(z)| e^{-\varepsilon \operatorname{Re} z^\alpha} = |F| e^{-\varepsilon r^\alpha \cos \alpha\theta}$

where $z = re^{i\theta}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Then $-\alpha\frac{\pi}{2} \le \alpha\theta \le \alpha\frac{\pi}{2}$ & $\alpha < 1$

$$\Rightarrow \exists \delta > 0 \text{ s.t. } \cos \alpha\theta \ge \underset{\cos\left(\frac{\alpha\pi}{2}\right)}{\overset{\shortparallel}{\delta}} > 0 \quad \forall \theta \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$

$$\begin{aligned}\Rightarrow \quad |G_\varepsilon(z)| &\le |F| e^{-\varepsilon\delta r^\alpha} \\ &\le C e^{cr^\gamma - \varepsilon\delta r^\alpha} \\ &= C e^{(c - \varepsilon\delta r^{\alpha-\gamma}) r^\gamma}\end{aligned}$$

since $\alpha > \gamma$, $\quad c - \varepsilon\delta r^{\alpha-\gamma} < 0$ for sufficiently large r

Hence $|G_\varepsilon(z)|$ is bounded and $\to 0$ as $|z| \to +\infty$.

$\Rightarrow$ max $|G_\varepsilon(z)|$ attained in a compact region

says $\{\mathrm{Re}\, z \geq 0\} \cap \{|z| \leq R\}$

By maximum principle, it happens on $\{\mathrm{Re}\, z = 0\}$ or $\{|z| = R\}$

If it is on $\{|z|=R\}$, it will be an interior max & implies

$G_\varepsilon(z) \equiv$ const. & hence $= 0$ (as $G_\varepsilon \to 0$ as $|z| \to \infty$)

$$\Rightarrow \quad F(z) \equiv 0 \quad . \quad \therefore \quad |F(z)| \leq 1 .$$

If it happens at $iy_0 \in \{\mathrm{Re}\, z = 0\}$,

then $\forall z \in \{\mathrm{Re}\, z \geq 0\}$

$$|G_\varepsilon(z)| \leq |F(iy_0) e^{-\varepsilon (iy_0)^\alpha}|$$

$$\leq 1 \cdot e^{-\varepsilon \mathrm{Re} (iy_0)^\alpha} = e^{-\varepsilon |y_0| \cos \frac{\alpha\pi}{2}}$$

$$\therefore \quad |F(z)| \leq e^{\varepsilon \mathrm{Re}(z^\alpha)} e^{-\varepsilon |y_0| \cos \frac{\alpha\pi}{2}} \quad \forall z \in \{\mathrm{Re}\, z \geq 0\}$$

Since $\varepsilon > 0$ is arbitrary, we have $|F(z)| \leq 1$.

\#

(2) (a) Let $a_n \in \mathbb{C} \setminus \{0\}$ be a seq. of complex number such that $\lim_{n\to\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\varepsilon}} < \infty$, $\forall \varepsilon > 0$

Show that the infinite product

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right),$$

defines an entire function of finite order. What is the order ρ_f of f?

(b) Let $g(z) = e^{z} f(z)$. Find the order ρ_g and the Hadamard factorization of g.

<u>Soln</u> (a) $\forall R > 0$, if $|z| < R$, then

$$\sum_{n=1}^{\infty} \left|\frac{z}{a_n}\right| = |z| \sum_{n=1}^{\infty} \frac{1}{|a_n|} \leq R \cdot \sum_{n=1}^{\infty} \frac{1}{|a_n|} \text{ convergent}$$

$\therefore$ $f(z)$ defines an entire function with zeros $\{a_n\}$

Now $\forall \varepsilon > 0$,

$$|f(z)| = \lim_{N\to\infty} \prod_{n=1}^{N} \left|1 - \frac{z}{a_n}\right| \leq \lim_{N\to\infty} \prod_{n=1}^{N} \left(1 + \frac{|z|}{|a_n|}\right)$$

$$\leq \lim_{N\to\infty} e^{\sum_{n=1}^{N} \log\left(1 + \frac{|z|}{|a_n|}\right)}$$

$\Rightarrow \forall |z| \geq 1,$

$$|f(z)| \leq \lim_{N\to\infty} e^{\sum_{n=1}^{N} C_\varepsilon \left(\frac{|z|}{|a_n|}\right)^\varepsilon} \quad \text{for some } C_\varepsilon > 0$$

$$\leq e^{C_\varepsilon \left(\sum_{n=1}^{\infty} \frac{1}{|a_n|^\varepsilon}\right) |z|^\varepsilon} \quad \text{as } \sum_{n=1}^{\infty} \frac{1}{|a_n|^\varepsilon} < \infty$$

$\therefore$ f has finite growth of order $\leq \varepsilon, \forall \varepsilon > 0$

Hence $\rho_f = 0$.

(b) $g(z) = e^z f(z)$

$\Rightarrow \forall \varepsilon > 0,$

$$|g(z)| \leq e^{|z|} e^{C'_\varepsilon |z|^\varepsilon} \quad \left(C'_\varepsilon = C_\varepsilon \sum_{n=1}^{\infty} \frac{1}{|a_n|^\varepsilon} > 0 \right)$$

$$\leq e^{C''_\varepsilon |z|} \quad \text{for some } C''_\varepsilon, \text{ provided } |z| \geq 1$$

$\Rightarrow \rho_g \leq 1.$

If $\rho_g < 1$, then $0 \leq \rho_g < 0+1$.

Hadamard Thm $\Rightarrow$

$$\left(e^z f(z) =\right) g(z) = e^b \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) = e^b f(z)$$

for some constant b as $\{a_n\}$ are all zeroes of g

which is a contradiction.

$\therefore \quad \rho_g = 1.$

And Hadamard factorization of g must have the form

$$g(z) = e^{az+b} \prod_{n=1}^{\infty} E_1\left(\frac{z}{a_n}\right)$$

$$= e^{az+b} \prod_{n=1}^{\infty} \left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}}$$

$$\therefore \quad g(z) = e^{az+b} \lim_{N\to\infty} \prod_{n=1}^{N} \left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}}$$

$$= e^{az+b} \lim_{N\to\infty} \left[\prod_{n=1}^{N} \left(1-\frac{z}{a_n}\right)\right] e^{\left(\sum_{n=1}^{N} \frac{1}{a_n}\right) z}$$

As $\prod_{n=1}^{N} \left(1-\frac{z}{a_n}\right)$ & $e^{\left(\sum_{n=1}^{N} \frac{1}{a_n}\right) z}$ converge locally uniformly and absolutely $\left(\text{since } \sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty\right)$,

$$g(z) = e^{az+b} \left[\prod_{n=1}^{\infty} \left(1-\frac{z}{a_n}\right)\right] e^{\left(\sum_{n=1}^{\infty} \frac{1}{a_n}\right) z}$$

$$= e^{\left(a+\sum_{n=1}^{\infty} \frac{1}{a_n}\right) z + b} f(z)$$

$$\therefore \quad b = 0 \quad \text{and} \quad a + \sum_{n=1}^{\infty} \frac{1}{a_n} = 1$$

Hence $g(z) = e^{\left(1-\sum_{n=1}^{\infty} \frac{1}{a_n}\right) z} \prod_{n=1}^{\infty} \left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}}$ $\#$

(3) Recall $\dfrac{1}{\Gamma(s)} = e^{\gamma s} s \prod (1+\frac{s}{n})e^{-\frac{s}{n}}$, where γ = Euler's constant

(a) Show that $\Gamma(s) = \lim_{n\to\infty} \dfrac{n^s n!}{s(s+1)\cdots(s+n)}$ for $s \neq 0,-1,-2,\cdots$

(b) Let $f:(0,\infty)\to(0,\infty)$ be a positive function satisfying

(i) $\log f(x)$ is convex on $(0,\infty)$

(ii) $f(x+1) = x f(x)$, $\forall x>0$

(iii) $f(1)=1$.

Show that $f(x) = \Gamma(x)$, $\forall x>0$.

[Hint: Apply convexity to $n<n+x\le n+1$ & $n-1<n<n+x$ to get

$(n-1)^x (n-1)! \le f(n+x) \le n^x (n-1)!$ for $x\in(0,1]$]

Pf of (a) By $\dfrac{1}{\Gamma(s)} = e^{\gamma s} s \prod_{n=1}^{\infty}(1+\frac{s}{n})e^{-\frac{s}{n}}$

$$= e^{\gamma s} s \lim_{n\to\infty} \prod_{k=1}^{n} (1+\frac{s}{k})e^{-\frac{s}{k}}$$

$$= e^{\gamma s} s \lim_{n\to\infty} \prod_{k=1}^{n} \frac{s+k}{k}\cdot e^{-\frac{s}{k}}$$

$$= \lim_{n\to\infty} e^{\gamma s}\cdot s\, \frac{(s+1)(s+2)\cdots(s+n)}{n!} e^{-s}e^{-\frac{s}{2}}\cdots e^{-\frac{s}{n}}$$

$$= \lim_{n\to\infty} \frac{s(s+1)\cdots(s+n)}{n!}\cdot e^{\gamma s-(1+\frac{1}{2}+\cdots+\frac{1}{n})s}$$

Using $\gamma = \lim_{n\to\infty}(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n)$, we have

$$\frac{1}{\Gamma(s)} = \lim_{n\to\infty} \frac{s(s+1)\cdots(s+n)}{n!} e^{[\gamma-(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n)]s} e^{-s\log n}$$

$$= \lim_{n\to\infty} \left[\frac{s(s+1)\cdots(s+n)}{n!\, n^s} e^{[\gamma-(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n)]s}\right]$$

$$= \lim_{n\to\infty} \frac{s(s+1)\cdots(s+n)}{n!\, n^s} \qquad \#$$

(b) By (ii), $f(x+n) = x(x+1)\cdots(x+n-1)f(x)$ $\forall\, n = 1,2,3\cdots$

same as $\Gamma(x+n) = x(x+1)\cdots(x+n-1)\Gamma(x)$

Hence, we only need to show $f(x) = \Gamma(x)$ for $x \in (0,1]$.

$\forall\, n = 1,2,3,\cdots \qquad n \le n+x \le n+1 \qquad$ for $x \in (0,1]$

Note that $\quad n+x = (1-x)n + x(n+1)$

$\therefore$ convexity $\Rightarrow \quad \log f(n+x) \le (1-x)\log f(n) + x \log f(n+1)$

Note that (ii) & (iii) $\Rightarrow$ $\forall\, n \ge 2$,

$$f(n) = (n-1)f(n-1) = \cdots = (n-1)!\,f(1) = (n-1)!$$

$$\therefore \quad \log f(n+x) \le x\left(\log n! - \log (n-1)!\right) + \log(n-1)!$$

$$= x\log n + \log(n-1)!$$

$$\Rightarrow \quad f(n+x) \le n^x (n-1)! \qquad \text{———} \quad (1)$$

Similarly for $x \in (0,1]$, $\quad n-1 < n < n+x$

and note that $\quad n = \left(1-\frac{1}{1+x}\right)(n-1) + \frac{1}{1+x}(n+x)$

$$\Rightarrow \quad \log f(n) \le \left(1-\frac{1}{1+x}\right)\log f(n-1) + \frac{1}{1+x}\log f(n+x)$$

$$\Rightarrow \quad \log f(n) - \log f(n-1) \le \frac{1}{1+x}\left[\log f(n+x) - \log f(n-1)\right]$$

$$\Rightarrow \quad (1+x)\left[\log f(n) - \log f(n-1)\right] \le \log f(n+x) - \log f(n-1)$$

$$\Rightarrow \quad x\left[\log f(n) - \log f(n-1)\right] + \log f(n) \le \log f(n+x)$$

$$\Rightarrow \quad x\left[\log (n-1)! - \log (n-2)!\right] + \log (n-1)! \le \log f(n+x)$$

$$\Rightarrow \quad x\log(n-1) + \log(n-1)! \le \log f(n+x)$$

$$\Rightarrow \quad (n-1)^x (n-1)! \le f(n+x) \quad \text{———} \quad (2)$$

By (ii) & inequalities (1) & (2), we have

$$\frac{(n-1)^x (n-1)!}{x(x+1)\cdots(x+n-1)} \le f(x) \le \frac{n^x (n-1)!}{x(x+1)\cdots(x+n-1)}, \quad \forall n$$

$$\le \frac{n^x\, n!}{x(x+1)\cdots(x+n)} \cdot \frac{x+n}{n}$$

By part (a), $\lim\limits_{n\to\infty} \dfrac{(n-1)^x (n-1)!}{x(x+1)\cdots(x+n-1)} = \Gamma(x)$, & $\lim\limits_{n\to\infty} \dfrac{x+n}{n} = 1$, we have $\Gamma(x) \le f(x) \le \Gamma(x) \quad \forall\ x \in (0,1]$.

Hence $f(x) = \Gamma(x), \ \forall\ x > 0$. ※