§2 The Schwarz Lemma; Automorphisms of the Disc and Upper Half-Plane

Lemma 2.1 (Schwarz Lemma)

If $f: \mathbb{D} \to \mathbb{D}$ holo & $f(0)=0$. Then

(i) $|f(z)| \leq |z| \quad \forall\, z \in \mathbb{D}$

(ii) If $\exists\, z_0 \in \mathbb{D}\setminus\{0\}$ s.t. $|f(z_0)| = |z_0|$, then f is a rotation.

(iii) $|f'(0)| \leq 1$, and if $|f'(0)| = 1$, then f is a rotation.

Pf: (i) Since $f(0)=0$, $\dfrac{f(z)}{z}$ is holo in $\mathbb{D}$.

For $|z| = r < 1$, we have $\left|\dfrac{f(z)}{z}\right| \leq \dfrac{1}{r}$ (Since $|f(z)| \leq 1$)

Maximum modulus principle $\Rightarrow \left|\dfrac{f(z)}{z}\right| \leq \dfrac{1}{r}, \ \forall\, |z| \leq r.$

Letting $r \to 1$, we have $\left|\dfrac{f(z)}{z}\right| \leq 1, \ \forall\, z \in \mathbb{D}.$

(ii) If $|f(z_0)| = |z_0|\ (\neq 0)$, then $\left|\dfrac{f(z_0)}{z_0}\right| = \max \left|\dfrac{f(z)}{z}\right|$

$\Rightarrow \left|\dfrac{f(z)}{z}\right|$ attains a maximum in the interior

$\Rightarrow \dfrac{f(z)}{z} = c$ is a constant with $|c| = 1 = \left|\dfrac{f(z_0)}{z_0}\right|$

$\therefore f(z) = e^{i\theta} z$ for some θ

(iii) Note that $f'(0) = \lim_{z\to 0} \frac{f(z)}{z}$ (as $f(0)=0$), (i) $\Rightarrow |f'(0)| \le 1$.

If $|f'(0)| = 1$, $\left|\frac{f(z)}{z}\right|$ attains maximum at $z=0$

$\Rightarrow f(z) = e^{i\theta} z$ as in (ii). $\#$

2.1 Automorphisms of the Disc

Def: • A conformal map from an open set Ω onto itself is called an automorphism of Ω.

• Set of all automorphisms of Ω, denoted by $Aut(\Omega)$, forms a group called the automorphism group of Ω.

Remarks: (i) clearly $Id_\Omega \in Aut(\Omega)$

(ii) The group operation of $Aut(\Omega)$ is composition of maps.

Egs: (i) Rotation $r_\theta : z \mapsto e^{i\theta} z \in Aut(\mathbb{D})$

with inverse the rotation $r_{-\theta} : z \mapsto e^{-i\theta} z$.

(ii) $\forall\, \alpha \in \mathbb{D}$, $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z} \in Aut(\mathbb{D})$ (Ex7 of Ch1 of the Textbook)

In fact, $\alpha \in \mathbb{D} \Leftrightarrow |\alpha| < 1 \Leftrightarrow \left|\frac{1}{\bar{\alpha}}\right| > 1$

$\therefore$ $\psi_\alpha(z)$ is holo. in $\mathbb{D}$,

And for $e^{i\theta} \in \partial D$,

$$|\psi_\alpha(e^{i\theta})| = \left|\frac{\alpha - e^{i\theta}}{1-\bar{\alpha}e^{i\theta}}\right| = \left|\frac{1}{e^{i\theta}} \cdot \frac{\alpha - e^{i\theta}}{e^{-i\theta} - \bar{\alpha}}\right| = \frac{|\alpha - e^{i\theta}|}{|\overline{e^{i\theta} - \alpha}|} = 1$$

Maximum principle $\Rightarrow$ $|\psi_\alpha(z)| < 1$, $\forall z \in D$. (as $\psi_\alpha \not\equiv$ const.)

$\therefore$ $\psi_\alpha : \mathbb{D} \to \mathbb{D}$.

Solving $w = \dfrac{\alpha - z}{1-\bar{\alpha}z}$ $\Rightarrow$ $z = \dfrac{\alpha - w}{1-\bar{\alpha}w} = \psi_\alpha(w)$

$\therefore$ ψ_α invertible & $\psi_\alpha^{-1} = \psi_\alpha$, hence conformal, ie $\psi_\alpha \in \mathrm{Aut}(\mathbb{D})$

<u>Thm 2.2</u> If $f \in \mathrm{Aut}(\mathbb{D})$, then $\exists\, \theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ s.t.

$$f(z) = e^{i\theta}\frac{\alpha - z}{1-\bar{\alpha}z}$$

Note: α is the unique zero of f in $\mathbb{D}$.

<u>Pf</u> Let $f \in \mathrm{Aut}(\mathbb{D})$, then f^{-1} exists $\in \mathrm{Aut}(\mathbb{D})$.

Hence $\alpha \overset{(\text{denote})}{=} f^{-1}(0) \in \mathbb{D}$.

Consider $g(z) = f \circ \psi_\alpha(z)$ (ψ_α as in Eg(ii) above)

Then $g \in \mathrm{Aut}(\mathbb{D})$ and

$$g(0) = f\circ\psi_\alpha(0) = f(\alpha) = 0$$

Schwarz Lemma $\Rightarrow$ $|g(z)| \leq |z|, \ \forall z \in \mathbb{D}$

On the other hand,

$$g^{-1}(0) = \psi_\alpha^{-1}\circ f^{-1}(0) = \psi_\alpha(\alpha) = 0$$

$\Rightarrow$ $|g^{-1}(w)| \leq |w|, \ \forall w \in \mathbb{D}$ (by Schwarz Lemma)

$\Rightarrow$ $|z| = |g^{-1}(g(z))| \leq |g(z)|, \ \forall z \in \mathbb{D}$

Therefore, $|z| = |g(z)| \ \forall z \in \mathbb{D}$

Schwarz Lemma again, $g(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$

i.e. $f\circ\psi_\alpha(z) = e^{i\theta} z$

$\Rightarrow$ $f(z) = f\circ\psi_\alpha(\psi_\alpha(z)) = e^{i\theta}\psi_\alpha(z) = e^{i\theta}\dfrac{\alpha - z}{1-\bar{\alpha}z}$. #

<u>Cor 2.3</u>: $\mathrm{Aut}_0(\mathbb{D}) \overset{\text{def}}{=} \{f \in \mathrm{Aut}(\mathbb{D}) : f(0) = 0\}$

$$= \{r_\theta : z \mapsto e^{i\theta} z, \ \theta \in \mathbb{R}\}$$

<u>Pf</u>: Putting $\alpha = 0$ in Thm 2.2.

<u>Remark</u>: $\mathrm{Aut}(\mathbb{D})$ <u>acts transitively</u> on $\mathbb{D}$ in the sense that $\forall z_0, z_1 \in \mathbb{D}$, $\exists f \in \mathrm{Aut}(\mathbb{D})$ such that

$$f(z_0) = z_1.$$

In fact $f = \psi_{z_1} \circ \psi_{z_0} \in \mathrm{Aut}(\mathbb{D})$ is the required map

since $\begin{cases} \psi_\alpha(0) = \alpha \ \& \\ \psi_\alpha(\alpha) = 0, \end{cases}$ (where ψ_α as in Eg(ii) above)

2.2 Automorphisms of the Upper Half-Plane

Recall: conformal maps

$$\begin{array}{ccc} F : \mathbb{H} & \to & \mathbb{D} \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ z & \longmapsto & \dfrac{i - z}{i + z} \end{array}$$

and it's inverse

$$\begin{array}{ccc} F^{-1} = G : \mathbb{D} & \longrightarrow & \mathbb{H} \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ w & \longmapsto & i\,\dfrac{1-w}{1+w} \end{array}$$

Define

$$\begin{array}{ccc} \Gamma : \mathrm{Aut}(\mathbb{D}) & \to & \mathrm{Aut}(\mathbb{H}) \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ \varphi & \longmapsto & F^{-1} \circ \varphi \circ F \end{array}$$

"conjugate by F"

Γ is clearly well-defined (by the figure & φ, F are conformal)

$$\begin{tikzcd} \mathbb{D} \arrow[r, "\varphi"] \arrow[d, "F^{-1}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{D} \arrow[d, "F^{-1}"] \\ \mathbb{H} \arrow[r, "\Gamma(\varphi)"'] \arrow[u, bend left, "F"] & \mathbb{H} \end{tikzcd}$$

Also $\Gamma^{-1} : \mathrm{Aut}(\mathbb{H}) \to \mathrm{Aut}(\mathbb{D})$ exists and

$\Gamma^{-1}(\psi) = F \circ \psi \circ F^{-1}$ "conjugate by F^{-1}".

$$\boxed{\Gamma : \mathrm{Aut}(\mathbb{D}) \to \mathrm{Aut}(\mathbb{H}) \text{ is a group isomaphism}}$$

Pf: We've seen that Γ is bijection, it remains to check that Γ is a group homomaphism: $\forall\ \varphi_1, \varphi_2 \in \mathrm{Aut}(\mathbb{D})$,

$$\begin{aligned} \Gamma(\varphi_1 \circ \varphi_2) &= F^{-1} \circ (\varphi_1 \circ \varphi_2) \circ F \\ &= F^{-1} \circ \varphi_1 \circ F \circ F^{-1} \circ \varphi_2 \circ F \\ &= \Gamma(\varphi_1) \circ \Gamma(\varphi_2) \end{aligned}$$

<u>Remark</u>: This fact can be generalized to any conformally equivalent open sets U and V.