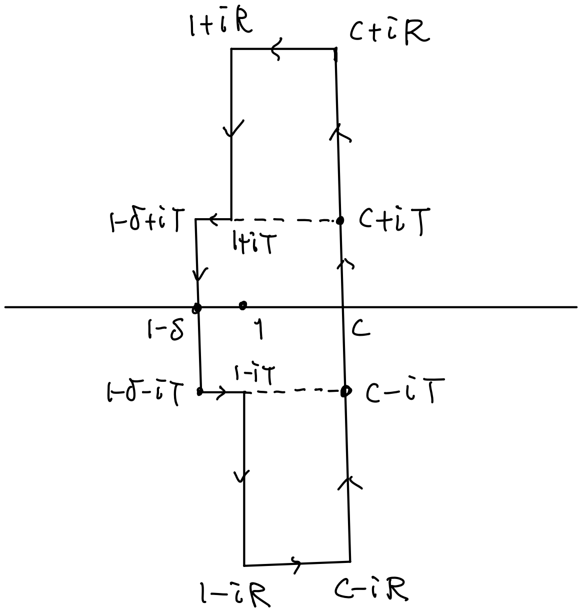
2.1 Proof of the asymptotics for ψ_1

(ie. Final step of the Proof of Prime Number Theorem)

Denote $F(s) = \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)$ where x fixed (& suff. large) say ≥ 2

Then Prop 2.3 $\Rightarrow$ $\forall c > 1$, $\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\,ds$

Consider the contour



$(R \to +\infty)$

where $T \geq 3$ and $\delta > 0$ is chosen (depending on T)

such that $\zeta(s) \neq 0$ along the contour.

This can be done since $\zeta(s) \neq 0$ $\forall$ $\mathrm{Re}(s) \geq 1$ (Thm 1.1 & 1.2)

By Prop 2.7(ii) in Ch6 and Prop 1.6 in Ch7,

$\forall \varepsilon > 0$, $\exists$ $C_\varepsilon > 0$ s.t

$$|\zeta'(s)| \leq C_\varepsilon |t|^\varepsilon \quad \text{and} \quad (\forall \sigma \geq 1 \ \& \ |t| \geq 1)$$

$$1/|\zeta(s)| \leq C_\varepsilon |t|^\varepsilon$$

Hence $\forall \eta > 0$, $\exists A > 0$ s.t.

$$(*), \quad \left| \frac{\zeta'(s)}{\zeta(s)} \right| \le A|t|^{\eta} \qquad \forall \sigma \ge 1 \ \& \ |t| \ge 1$$

$\Rightarrow$ For $R\,(> T)$ sufficiently large, and

$s \in [1+iR, c+iR]$ or $[1-iR, c-iR]$, (top and bottom horizontal line segments)

$$|F(s)| = \frac{|x^{s+1}|}{|s(s+1)|} \left| \frac{\zeta'(s)}{\zeta(s)} \right|$$

$$\le x^{c+1} \cdot \frac{1}{|s(s+1)|} \left| \frac{\zeta'(s)}{\zeta(s)} \right| \le A' |t|^{-2+\eta}$$ for some $A' > 0$ (A' indep. of s)

$$\Rightarrow \left| \int_{c+iR}^{1+iR} F(s)\,ds \right| \le A' R^{-2+\eta} (c-1) \to 0 \text{ as } R \to \infty$$

and $$\left| \int_{1-iR}^{c-iR} F(s)\,ds \right| \le A' R^{-2+\eta} (c-1) \to 0 \text{ as } R \to \infty$$

Letting $R \to \infty$, Residue formula $\Rightarrow$

$$\operatorname{res}_{s=1} F(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\,ds - \left[\begin{array}{l} \displaystyle \frac{1}{2\pi i} \left(\int_{1+iT}^{1+i\infty} + \int_{1-i\infty}^{1-iT} \right) F(s)\,ds \\ \displaystyle + \frac{1}{2\pi i} \left(\int_{1-\delta+iT}^{1+iT} - \int_{1-\delta-iT}^{1-iT} \right) F(s)\,ds \\ \displaystyle + \frac{1}{2\pi i} \int_{1-\delta-iT}^{1-\delta+iT} F(s)\,ds \end{array} \right]$$

By Cor 2.6 of Ch6,

$$\zeta(s) = \frac{1}{s-1} + H(s) \quad \text{near } s=1 \text{ with holo } H(s).$$

$$\Rightarrow \quad -\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \underbrace{\frac{H(s)+(s-1)H'(s)}{1+(s-1)H(s)}}_{\text{holo. near } s=1}$$

$$\therefore \quad \operatorname{res}_{s=1} F(s) = \operatorname{res}_{s=1}\left[\frac{x^{s+1}}{s(s+1)}\cdot\left(\frac{1}{s-1} - h(s)\right)\right]$$

$$= \frac{x^2}{2}.$$

$$\therefore \quad \psi_1(x) = \frac{x^2}{2} + \frac{1}{2\pi i}\left(\int_{1+iT}^{1+i\infty} + \int_{1-i\infty}^{1-iT}\right)F(s)\,ds$$

$$+ \frac{1}{2\pi i}\left(\int_{1-\delta+iT}^{1+iT} - \int_{1-\delta-iT}^{1-iT}\right)F(s)\,ds + \frac{1}{2\pi i}\int_{1-\delta-iT}^{1-\delta+iT} F(s)\,ds$$

Since we care only the limit as $x \to +\infty$, we may assume $x \geq 2$ in our estimates.

(i) $$\left|\int_{1+iT}^{1+i\infty} F(s)\,ds\right| \leq \int_T^{\infty} \frac{|x^{2+it}|}{|(1+it)(2+it)|}\left|\frac{\zeta'(1+it)}{\zeta(1+it)}\right| dt$$

$$\leq x^2 \cdot \int_T^{\infty} \frac{1}{|1+it||2+it|}\cdot A|t|^{1/2}\,dt \qquad (\text{take } \eta = \tfrac{1}{2} \text{ in } (*)_1)$$

Clearly the integral converges and hence

$\forall \varepsilon > 0$, $$\left|\frac{1}{2\pi i}\int_{1+iT}^{1+i\infty} F(s)\,ds\right| \leq \varepsilon\frac{x^2}{2} \qquad \text{for suff. large } T.$$

Same argument $\Rightarrow$

$$\forall \varepsilon > 0, \qquad \left| \frac{1}{2\pi i} \int_{1-i\infty}^{1-iT} F(s)\,ds \right| \le \varepsilon \frac{X^2}{2} \qquad \text{for suff. large } T.$$

(ii)
$$\left| \frac{1}{2\pi i} \int_{1-\delta+iT}^{1+iT} F(s)\,ds \right| \le \frac{1}{2\pi} \int_{1-\delta}^{1} \frac{|X^{1+\sigma+iT}|}{|\sigma+iT|\,|\sigma+1+iT|} \left| \frac{\zeta'(\sigma+iT)}{\zeta(\sigma+iT)} \right| d\sigma$$

$$\le \frac{1}{2\pi} \int_{1-\delta}^{1} \frac{X^{1+\sigma}}{T^2} A T^{\frac{1}{2}}\, d\sigma \qquad (\eta = \tfrac{1}{2} \text{ in } (*),)$$

$$= C_T' \int_{1-\delta}^{1} X^{1+\sigma}\, d\sigma = C_T' \int_{1-\delta}^{1} e^{(1+\sigma)\log X}\, d\sigma$$

$$= C_T' \left[\frac{e^{(1+\sigma)\log X}}{\log x} \right]_{1-\delta}^{1} \le C_T' \frac{X^2}{\log x}$$

$(X \ge 2 \Rightarrow \log X \ge \log 2 > 0)$

Similarly $\left| \frac{1}{2\pi i} \int_{1-\delta-iT}^{1-iT} F(s)\,ds \right| \le C_T' \frac{X^2}{\log X}$ for same const. $C_T' > 0$.

(iii)
$$\left| \frac{1}{2\pi i} \int_{1-\delta-iT}^{1-\delta+iT} F(s)\,ds \right| \le \frac{1}{2\pi} \int_{-T}^{T} \frac{|X^{1+(1-\delta)+it}|}{|1-\delta+it|\,|2-\delta+it|} \left| \frac{\zeta'(1-\delta+it)}{\zeta(1-\delta+it)} \right| dt$$

$$\le \frac{1}{2\pi} \int_{-T}^{T} \frac{X^{2-\delta}}{|1-\delta+it|\,|2-\delta+it|} \left| \frac{\zeta'(1-\delta+it)}{\zeta(1-\delta+it)} \right| dt$$

$$\le C_T X^{2-\delta} \qquad \text{for some const. } C_T$$

(depending on T, δ and hence depending on T as δ is chosen according to T)

Hence (i), (ii) & (iii) $\to$ $\forall \varepsilon > 0$, $\exists\, \delta > 0$, C_T & C_T' s.t.

$$\left| \psi_1(x) - \frac{x^2}{2} \right| \le 2\varepsilon \frac{x^2}{2} + C_T' \frac{x^2}{\log x} + C_T x^{2-\delta}$$ for sufficiently large T

$$\Rightarrow \quad \left| \frac{2\psi_1(x)}{x^2} - 1 \right| \le 2\varepsilon + \underbrace{2C_T' \frac{1}{\log x} + 2C_T \frac{1}{x^\delta}}_{\longrightarrow\, 0 \text{ as } x \to \infty}.$$

Hence $\left| \frac{2\psi_1(x)}{x^2} - 1 \right| \le 4\varepsilon$ for sufficiently large x.

$$\Rightarrow \quad \lim_{x \to +\infty} \frac{2\psi_1(x)}{x^2} = 1$$

i.e. $\psi_1(x) \sim \frac{x^2}{2}$ as $x \to \infty$.

This completes the proof of the prime number theorem. #

Ch 8 Conformal Mappings

§1 Conformal Equivalence and Examples

Def • A bijective holomorphic function $f: U \to V$ (U, V open in $\mathbb{C}$) is called a conformal map or biholomorphism.

• In this case, U and V are said to be conformally equivalent or simply biholomorphic.

Prop 1.1 • If $f: U \to V$ is holomorphic and injective, then

$$f'(z) \neq 0 \quad \forall z \in U.$$

• In particular, $f^{-1}: f(U) \to U$ is holomorphic

($\Rightarrow$ inverse of conformal map is also holomorphic and hence conformal)

Remarks: • Prop 1.1 $\Rightarrow$

U & V are conformally equivalent

$\Leftrightarrow \exists$ holo. $f: U \to V$ and holo $g: V \to U$

s.t. $g(f(z)) = z \quad \forall z \in U$ &

$f(g(w)) = w \quad \forall w \in V$.

• Some authors call a holomorphic map $f: U \to V$ conformal if $f'(z) \neq 0$, $\forall z \in U$, not necessary bijective (globally)

(In this course, we'll follow Textbook's convention.)

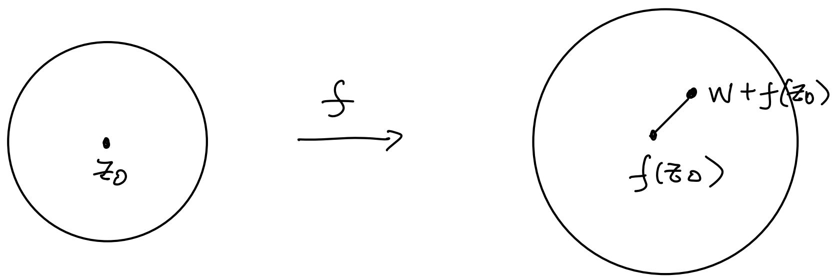
Pf of Prop 1.1

Suppose on the contrary that $f'(z_0)=0$ for some $z_0 \in U$.

Then $f(z)-f(z_0) = a(z-z_0)^k + G(z)$ near z_0

where $a \neq 0$, $k \geq 2$ and

G vanishing to order $k+1$ at z_0. $\left(\text{i.e. } \dfrac{|G(z)|}{|z-z_0|^{k+1}} \leq C \text{ near } z_0 \right)$



Consider $w \neq 0$ with $|w|$ sufficiently small

Then $f(z)-(f(z_0)+w) = [a(z-z_0)^k - w] + G(z)$

$= F(z) + G(z)$

where $F(z) = a(z-z_0)^k - w$.

Then
- $|F(z)| > |G(z)|$ on a sufficiently small circle centered at z_0
- $k \geq 2 \Rightarrow F(z)$ has at least 2 zeros inside that circle.

Rouché's Thm $\Rightarrow$ $f(z)-(f(z_0)+w)$ has at least 2 zeros there too.

Since f' is holo & hence z_0 is an isolated zero.

We may assume, by choosing a smaller circle, that

$f'(z) \neq 0$ $\forall z$ insides the circle except $z=z_0$.

$\Rightarrow$ the zeros of $f(z) - (f(z_0) + w)$ are distinct

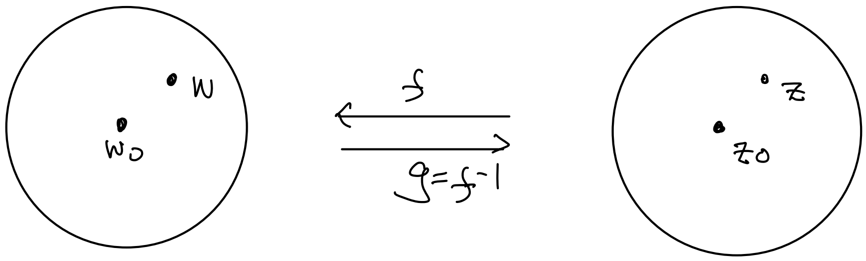
$\therefore$ f is not injective near z_0.

This proves the 1st statement.

For the 2nd statement, let $g = f^{-1} : f(U) \to U$

(Open mapping theorem (Thm 4.4, Ch 3) $\Rightarrow$ g is continuous)

Suppose that $w_0 \in f(U)$ and w close to w_0, but $w \neq w_0$.



Then $\exists$ z & $z_0 \in U$ s.t. $w = f(z)$ & $w_0 = f(z_0)$.

Hence

$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{\left(\dfrac{f(z) - f(z_0)}{z - z_0}\right)}$$

Since $f'(z_0) \neq 0$, we have

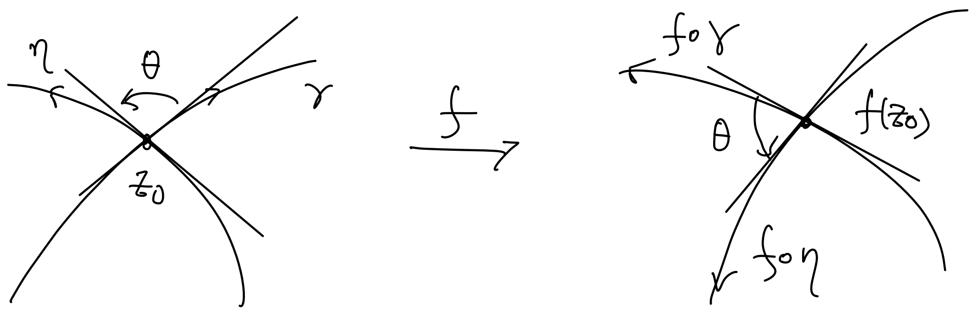
$$\lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{\lim\limits_{z \to z_0}\left(\dfrac{f(z) - f(z_0)}{z - z_0}\right)} = \frac{1}{f'(z_0)} \quad \text{exists}$$

$\therefore$ g is holo. and $g'(w_0) = \dfrac{1}{f'(g(w_0))}$ ※

Remark: $\boxed{\begin{array}{l}\text{If } f: U \to \mathbb{C},\ z_0 \in U, \text{ and } f'(z_0) \neq 0. \\ \text{Then } f \text{ preserves angles at } z_0.\end{array}}$

The precise formulation is:

Let γ & η be two (smooth oriented) curves intersecting at z_0, then the angle from the curve $f \circ \gamma$ to the curve $f \circ \eta$ at $f(z_0)$ equals the angle from the curve γ to the curve η at z_0.



(Problem 2 on page 255 of the Textbook.)

Hence

$\boxed{\text{conformal maps preserve angles}}$

1.1 The disc and upper half-plane

Notations:
- (unit) Disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$
- upper half-plane $\mathbb{H} = \{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}$

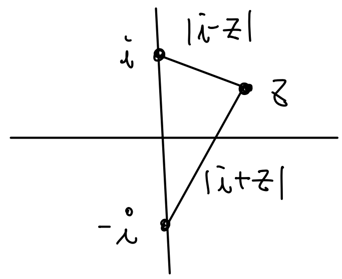
Thm 1.2: The map
$$F : \mathbb{H} \to \mathbb{D}, \quad z \mapsto \frac{i-z}{i+z}$$
is a conformal map

with inverse
$$G = F^{-1} : \mathbb{D} \to \mathbb{H}, \quad w \mapsto i\,\frac{1-w}{1+w}.$$

Pf: Clearly
$$\begin{cases} z \in \mathbb{H} \Rightarrow i+z \neq 0 \Rightarrow F \text{ is holo.} \\ w \in \mathbb{D} \Rightarrow 1+w \neq 0 \Rightarrow G \text{ is holo.} \end{cases}$$

Then

- $$|F(z)| = \left|\frac{i-z}{i+z}\right| < 1$$
$$\Rightarrow \quad F(\mathbb{H}) \subset \mathbb{D},$$



- And for $w = u + iv \in \mathbb{D}$,
$$\operatorname{Im}(G(w)) = \operatorname{Im}\left(i\,\frac{1-u-iv}{1+u+iv}\right)$$
$$= \frac{1-u^2-v^2}{(1+u)^2+v^2} > 0$$
$$\therefore \quad G(\mathbb{D}) \subset \mathbb{H}.$$

Finally
$$F(G(w)) = \frac{i - i\,\frac{1-w}{1+w}}{i + i\,\frac{1-w}{1+w}} = w$$

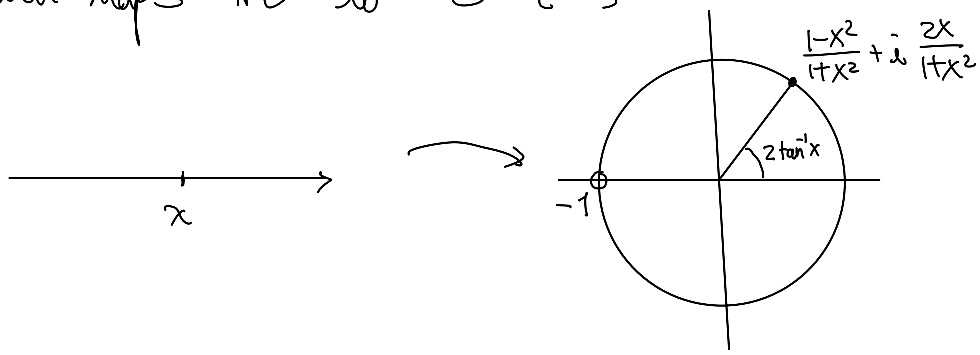
&
$$G(F(z)) = i \cdot \frac{1 - \frac{i-z}{i+z}}{1 + \frac{i-z}{i+z}} = z$$

#

<u>Remark</u>: $F\big|_{\partial \mathbb{H}} : \mathbb{R} \to \partial \mathbb{D} = S^1$ is continuous, and

$$F(x) = \frac{i-x}{i+x} = \frac{1-x^2}{1+x^2} + i\,\frac{2x}{1+x^2}$$

which maps $\mathbb{R}$ to $S^1 \setminus \{-1\}$



One should think of $F(\infty) = -1$. And $G(-1) = \infty$.

<u>Def</u>: Mappings of the form

$$z \mapsto \frac{az+b}{cz+d}, \qquad \left(\mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\} \right)$$

with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$,

are called <u>fractional linear transformations</u>

<u>Remarks</u>: (i) $ad - bc \neq 0 \iff cz + d \neq k(az+b)$ and $(az+b) \neq k(cz+d)$

(for some $k \in \mathbb{C}$)

$\iff z \mapsto \frac{az+b}{cz+d}$ is not a constant map.

(ii) Some other authors call them <u>linear fractional transformations</u>, or <u>Möbius transformations</u>.

1.2 Further examples

Eg 1 • Translations are conformal

$$z \mapsto z+h : \mathbb{C} \to \mathbb{C} \qquad (h \in \mathbb{C})$$

(Inverse $w \mapsto w-h$)

Remark: If $h \in \mathbb{R}$, then $z \mapsto z+h : \mathbb{H} \to \mathbb{H}$ is conformal.

• Dilations are conformal

$$z \mapsto cz : \mathbb{C} \to \mathbb{C} \qquad (c \in \mathbb{C} \setminus \{0\})$$

(Inverse $w \mapsto c^{-1}w$)

Remarks:

(i) If $|c|=1$, then $c = e^{i\varphi}$ and

$$z \mapsto cz = e^{i\varphi} z : \mathbb{C} \to \mathbb{C} \text{ is a rotation}$$

When restricted, $z \mapsto e^{i\varphi} z : \mathbb{D} \to \mathbb{D}$ is also conformal.

(ii) $c > 0$: $z \mapsto cz$ is a (real) dilation

when restricted, $z \mapsto cz : \mathbb{H} \to \mathbb{H}$ is conformal

(iii) $c < 0$: $z \mapsto cz = -|c|z$

is a (real) dilation by $|c|$ followed by

a rotation of angle π.

Note that translations and dilations are special cases of fractional linear transformations :

translations:

$$z \mapsto z+h = \frac{z+h}{0\cdot z+1} \quad \text{i.e. } a=1=d,\ b=h,\ c=0$$

$$\& \quad ad-bc = 1 \neq 0.$$

dilations $(c \neq 0)$

$$z \mapsto cz = \frac{cz+0}{0\cdot z+1} \quad \& \quad c\cdot 1 - 0\cdot 0 = c \neq 0.$$

Eg 1' (not in textbook)

(Complex) Inversion

$$z \mapsto \begin{cases} \frac{1}{z}, & \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\} \\ 0, & z=\infty \\ \infty, & z=0 \end{cases}$$

is conformal.

Note that Inversion is also a fractional linear transformation

$$z \mapsto \frac{1}{z} = \frac{0\cdot z+1}{z+0} \quad \& \quad 0\cdot 0 - 1\cdot 1 = -1 \neq 0.$$

Properties of fractional linear transformations

(1) Conformal as maps from $\mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$, hence angles preserving.

(2) f, g are fractional linear transformations $\Rightarrow$ $f\circ g$ is a fractional linear transformation.

(3) fractional linear transformation is a composition of translations, dilations and inversions.

(4) fractional linear transformations map "straight lines & circles" to "straight lines or circles".

Pf: (1) Clearly $f(z)=\dfrac{az+b}{cz+d}$ has derivatives

$$f'(z)=\frac{ad-bc}{(cz+d)^2}\neq 0 \quad \text{for } z\neq -\frac{d}{c}.$$

(we omit the discussion at $z=-\frac{d}{c}$ and $z=\infty$)

Also, clearly $g(w)=\dfrac{dw-b}{-cw+a}$ is the inverse of f

(Note: $z=-\frac{d}{c} \leftrightarrow w=\infty$, $z=\infty \leftrightarrow w=\frac{a}{c}$)

$\therefore$ f is conformal (from $\mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$)

(2) If $f(z)=\dfrac{az+b}{cz+d}$, $ad-bc\neq 0$

$g(z)=\dfrac{kz+l}{mz+n}$, $kn-lm\neq 0$

Then $f\circ g(z) = \dfrac{a\left(\frac{kz+l}{mz+n}\right)+b}{c\left(\frac{kz+l}{mz+n}\right)+d} = \dfrac{(ak+bm)z+(al+bn)}{(ck+dm)z+(cl+dn)}$

Note that $\begin{pmatrix} ak+bm & al+bn \\ ck+dm & cl+dn \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} k & l \\ m & n \end{pmatrix}$

$\therefore$ $(ak+bm)(cl+dn) - (al+bm)(ck+dm)$

$$= \det\begin{pmatrix} ak+bm & al+bn \\ ck+dm & cl+dn \end{pmatrix}$$

$$= \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \det\begin{pmatrix} k & l \\ m & n \end{pmatrix}$$

$$= (ad-bc)(kn-lm) \neq 0 .$$

$\therefore$ $f\circ g$ is a fractional linear transformation.

(3) $f(z) = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$

If $c=0$, then $d\neq 0$ & $f(z) = \left(\frac{a}{d}\right)z + \left(\frac{b}{d}\right)$

i.e. $z \longmapsto \underbrace{\left(\frac{a}{d}\right)z}_{\uparrow \text{ dilation } (a\neq 0)} \longmapsto \underbrace{\left[\left(\frac{a}{d}\right)z\right] + \left(\frac{b}{d}\right)}_{\uparrow \text{ translation}} = f(z)$

If $c\neq 0$, then

$$f(z) = \frac{az+b}{cz+d} = \frac{1}{c}\cdot\frac{az+b}{z+\frac{d}{c}}$$

$$= \frac{1}{c}\left[\frac{a(z+\frac{d}{c}) - \frac{ad}{c} + b}{z+\frac{d}{c}}\right]$$

$$= \frac{1}{c}\left[a - \frac{\frac{ad}{c} - b}{z+\frac{d}{c}}\right]$$

$$= \frac{a}{c} - \frac{(ad-bc)}{c^2}\cdot\frac{1}{z+\frac{d}{c}}$$

i.e. $\underset{\text{translation}}{z \mapsto z+\frac{d}{c}} \underset{\text{inversion}}{\to} \frac{1}{z+\frac{d}{c}} \underset{\text{dilation}}{\mapsto} -\frac{(ad-bc)}{c^2}\frac{1}{z+\frac{d}{c}}$

$\underset{\text{translation.}}{\mapsto} \frac{a}{c} - \frac{ad-bc}{c^2}\frac{1}{z+\frac{d}{c}}$

(4) Note that translations and dilations map straight lines to straight lines, and circles to circles.

Then because of (3), we only need to prove (4) for inversion $z \mapsto \frac{1}{z}$.

let $z = x+iy$ & $w = s+it = \frac{1}{z}$

then $s+it = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

i.e. $\begin{cases} s = \frac{x}{x^2+y^2} \\ t = -\frac{y}{x^2+y^2} \end{cases}$ (Inversion as a mapping from $\mathbb{R}^2\setminus\{0\} \to \mathbb{R}^2\setminus\{0\}$)

Also $wz=1 \Rightarrow |w|^2|z|^2=1$, $\Rightarrow \begin{cases} x = \dfrac{s}{s^2+t^2} \\ y = \dfrac{-t}{s^2+t^2} \end{cases}$

(ie. $s^2+t^2 = \dfrac{1}{x^2+y^2}$)

Now let $L: ax+by+c=0$ be a straight line

Then $$\frac{as}{s^2+t^2} - \frac{bt}{s^2+t^2} + c = 0$$

ie. $$c(s^2+t^2)+as-bt=0$$

If $c=0$ (ie. L passing thro the origin),

the image of L is the straight line

$$L': as-bt=0 \quad (\text{in } (s,t)\text{-plane}).$$

If $c\neq 0$ (ie. L not passing thro the origin)

$\therefore$ the image of L is the circle

$$C': s^2+t^2+\left(\frac{a}{c}\right)s-\frac{b}{c}t=0 \quad (\text{in } (s,t)\text{-plane})$$

Now let $C: x^2+y^2+ax+by+c=0$ be a circle.

Then we have $$\frac{1}{s^2+t^2}+\frac{as}{s^2+t^2}-\frac{bt}{s^2+t^2}+c=0$$

$$\Rightarrow c(s^2+t^2)+as-bt+1=0.$$

If $c=0$, the image of C is a straight line

$$L': as-bt+1=0$$

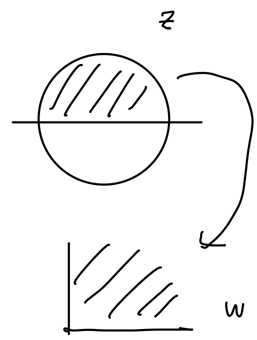
If $c\neq 0$, the image of C is a circle

$$C': s^2+t^2+\left(\frac{a}{c}\right)s-\left(\frac{b}{c}\right)t+\frac{1}{c}=0. \quad \#$$

Eg 3 (of the Text book)

$f(z) = \dfrac{1+z}{1-z} : \{ z = x+iy : |z|<1 \text{ and } y>0 \} = \mathbb{D}^+$

$\rightarrow \{ w = u+iv = u>0 \text{ and } v>0 \} = S$



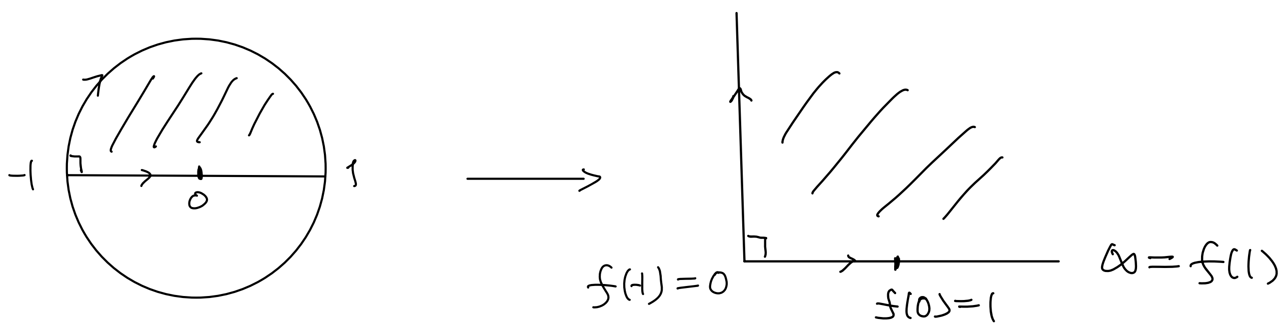
is conformal.

Note: f is a fractional linear transformation

$f(z) = \dfrac{z+1}{-z+1}$ with $1 \cdot 1 - (1)(-1) = 2 \neq 0$

$\therefore$ f is injective, hence remain to show $f(\mathbb{D}^+) = S$.

Observe that $f(-1) = 0$, $f(0) = 1$, $f(1) = \infty$



By property (4) of fractional linear transformation,

the real line segment between -1 & 1

maps to part of a straight line or a circle.

Since it passes through $f(-1) = 0$, $f(0) = 1$ & $f(1) = \infty$,

it is the positive real axis.

Similarly, the upper semi-circle maps to part of a straight line or a circle passing through 0 and ∞, and hence must be a straight line.

Since the angles from $[-1,1]$ to the semi-circle is $\frac{\pi}{2}$, the angle from the positive x-axis to the image straight line of the semi-circle is also $\frac{\pi}{2}$ (f conformal)

$\therefore$ the image of the upper semi-circle is the positive y-axis.

(Positivity can also be confirmed by $f(i) = \frac{1+i}{1-i} = i$)

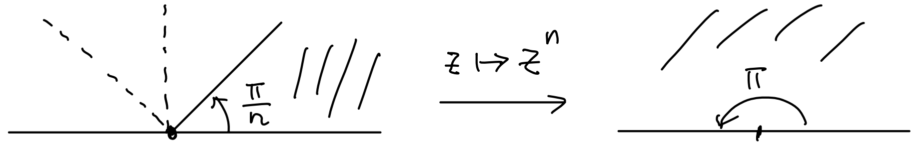
This shows that $f(D^+) = S$ (as f is conformal : $\mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$)

(Of course, all these can be proved by using coordinates as in the Textbook)

<u>Eg2</u> (Of the Textbook)

For $n = 1, 2, 3, \cdots$, $z \mapsto z^n : S \to \mathbb{H}$ is conformal,

where $S = \{ z \in \mathbb{C} : 0 < \arg(z) < \frac{\pi}{n} \}$

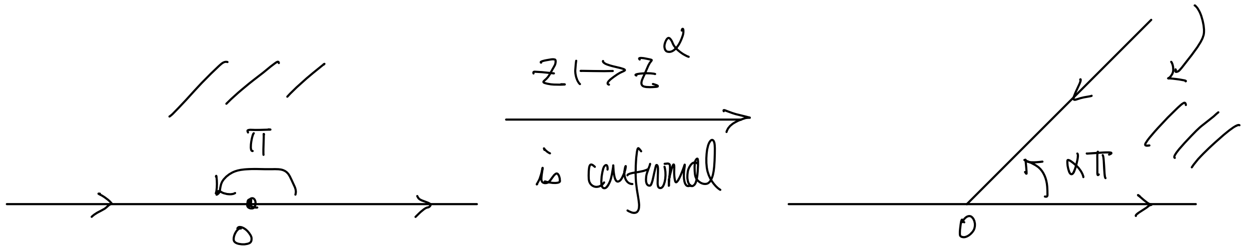


Inverse $w \mapsto w^{\frac{1}{n}} : \mathbb{H} \to S$

where $w^{\frac{1}{n}} = e^{\frac{1}{n}\log w}$ with $\log w$ = principal branch

More generally, for $0 < \alpha < 2$ $\left(0 < \frac{1}{n} \leq 2\right)$

$z \in \mathbb{H}$ $\qquad\qquad$ $S = \{ w \in \mathbb{C} : 0 < \arg(w) < \alpha\pi \}$



with inverse $\quad w \mapsto w^{\frac{1}{\alpha}} = e^{\frac{1}{\alpha}\log w}$

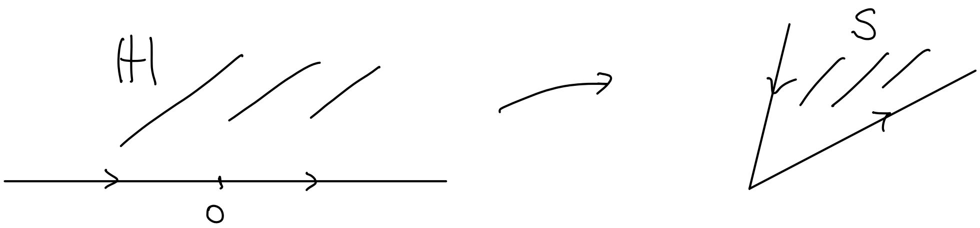
where branch of $\log w$ s.t. $0 < \arg w < \alpha\pi$.

(Boundary behavior as in the figure.)

Conclusion :

One can map $\mathbb{H}$ conformally to any (infinite) sector in $\mathbb{C}$

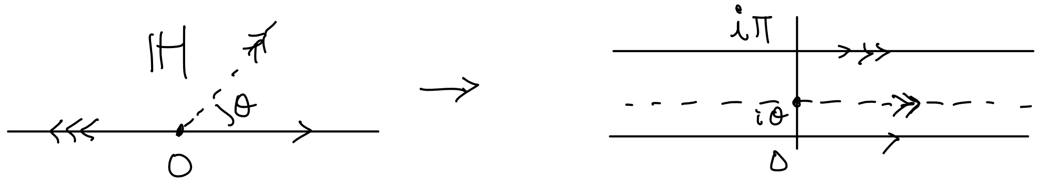
(by composing the maps here with translations & rotations.)



Eg 4 : $\quad z \mapsto \log z \quad$ branch defined by deleting $\{x < 0\}$

(i.e. $-\pi < \arg z < \pi$)

maps $\mathbb{H}$ conformally to strip $\{ w = u + iv : 0 < v < \pi \}$.



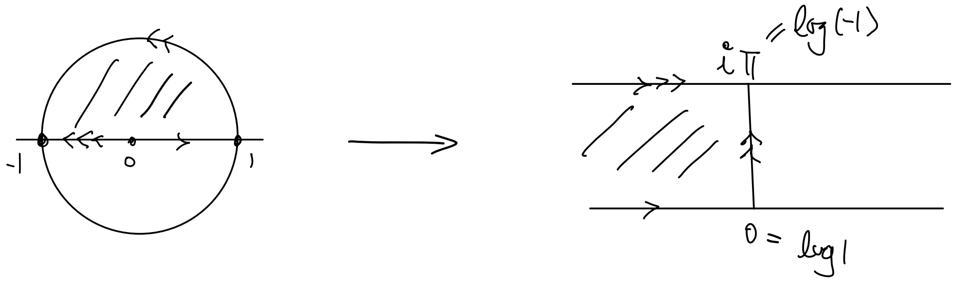
By the choice of the branch, for $z = re^{i\theta}$, $0 < \theta < \pi$

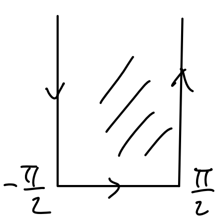
$$\log z = \log r + i\theta$$

$\therefore \quad u = \log r \in \mathbb{R}$ and $v = \theta \in (0, \pi)$

The inverse is $w \mapsto e^{w}$.

<u>Eg5</u> Same $z \mapsto \log z$ maps $\mathbb{D}^+ = \{ z = x+iy : |z| < 1, y > 0 \}$ conformally to <u>half strip</u> $\{ w = u + iv : u < 0, 0 < v < \pi \}$, since $u = \log r < 0$.



<u>Eg6</u>: $f(z) = e^{iz}$ maps  conformally onto