# Ch6 The Gamma and Zeta Functions

## §1 The Gamma Function

**Def**: For $s>0$, the *gamma function* is defined by

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1}\,dt. \qquad (1)$$

**Remark**: By definition,

$$\int_0^{\infty} e^{-t} t^{s-1}\,dt = \lim_{\varepsilon\to 0} \int_{\varepsilon}^{\frac{1}{\varepsilon}} e^{-t} t^{s-1}\,dt$$

Clearly $\Gamma(s)$ is well-defined for $s>0$:

$$\int_{\varepsilon}^{1} t^{s-1}\,dt = \frac{t^s}{s}\Big|_{\varepsilon}^{1} \to \frac{1}{s} \quad \text{as } \varepsilon\to 0.$$

(and $e^{-t}$ bdd near $t=0$ & rapidly decay as $t\to+\infty$)

**Prop 1.1** The formula $\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1}\,dt$ extends the domain of definition of $\Gamma(s)$ to the open half-plane

$\{s\in\mathbb{C} : \operatorname{Re}(s) > 0\}$.

Pf: It suffices to show that (1) defines a holomorphic function in

$$S_{\delta,M} = \{\delta < \operatorname{Re}(s) < M\}, \quad \forall\, 0<\delta<M<\infty.$$

Note that $t^{s-1} = e^{(s-1)\log t}$ is holomorphic in $s \in \mathbb{C}$

and hence $e^{-t} t^{s-1}$ is holo. in $s$, and continuous

in $(t, s) \in [\varepsilon, \frac{1}{\varepsilon}] \times [\delta, M]$.

Hence Thm 5.4 of Ch 2 $\Rightarrow$

$\forall \varepsilon > 0$, $\quad F_\varepsilon(s) = \int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t} t^{s-1} dt$ is holo. on $S_{\delta, M}$.

Note also that $\delta < Re(s) < M$

$$\Rightarrow \quad |e^{-t} t^{s-1}| = e^{-t} |e^{(s-1)\log t}| = e^{-t} e^{(Re(s)-1)\log t}$$
$$= e^{-t} t^{Re(s)-1}$$

(for $\varepsilon < 1$)
$$\Rightarrow \quad \int_0^\varepsilon |e^{-t} t^{s-1}| dt = \int_0^\varepsilon e^{-t} t^{Re(s)-1} dt$$
$$\leq \int_0^\varepsilon e^{-t} t^{\delta-1} dt \leq \frac{\varepsilon^\delta}{\delta}$$

$\therefore \int_0^\varepsilon e^{-t} t^{s-1} dt$ is convergent. (as improper integral near 0)

Similarly $\int_{\frac{1}{\varepsilon}}^\infty |e^{-t} t^{s-1}| dt \leq \int_{\frac{1}{\varepsilon}}^\infty e^{-t} t^{M-1} dt$

$$\leq C \int_{\frac{1}{\varepsilon}}^\infty e^{-\frac{t}{2}} dt \quad \text{(for some } C > 0\text{)}$$

$$\leq 2C e^{-\frac{1}{2\varepsilon}}$$

$\therefore \quad \int_{\frac{1}{\varepsilon}}^{\infty} e^{-t} t^{s-1}\, dt$ is also convergent.

Hence $\int_0^{\infty} e^{-t} t^{s-1}\, dt$ defined &

$$\int_0^{\infty} e^{-t} t^{s-1}\, dt - F_\varepsilon(s) = \int_0^{\varepsilon} e^{-t} t^{s-1}\, dt + \int_{\frac{1}{\varepsilon}}^{\infty} e^{-t} t^{s-1}\, dt$$

$$\Rightarrow \left| \int_0^{\infty} e^{-t} t^{s-1}\, dt - F_\varepsilon(s) \right| \leq \int_0^{\varepsilon} e^{-t} t^{Re(s)-1}\, dt + \int_{\frac{1}{\varepsilon}}^{\infty} e^{-t} t^{Re(s)-1}\, dt$$

$$\leq \frac{\varepsilon^\delta}{\delta} + 2Ce^{-\frac{1}{2\varepsilon}} \to 0 \quad \text{as } \varepsilon \to 0.$$

$\therefore$ (Thm 5.2 of Ch2) $\int_0^{\infty} e^{-t} t^{s-1}\, dt$ is a uniform limit of sequence of holo. function on $S_{\delta,M}$, hence

$\int_0^{\infty} e^{-t} t^{s-1}\, dt$ is a holo. on $S_{\delta,M}$, $\forall\ 0<\delta<M<\infty$

$\Rightarrow \int_0^{\infty} e^{-t} t^{s-1}\, dt$ define a holo on $\{Re(s) > 0\}$.

Clearly, when restricted to real $s>0$, $\int_0^{\infty} e^{-t} t^{s-1}\, dt = \Gamma(s)$,

$\therefore \int_0^{\infty} e^{-t} t^{s-1}\, dt$ is the required extension.

$\divideontimes$