**MATH4050 Real Analysis**
**Assignment 8**

There are 6 questions in this assignment. The page number and question number for each question correspond to that in Royden's Real Analysis, 3rd or 4th edition.

thus, wlg. $f_n \in \mathbb{R}\ \forall n$ & $f_n(x), f(x) \in \mathbb{R}$ a.e

1. (3rd: P.89, Q9; 4th: P.84, Q22)
   Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $(-\infty, +\infty)$ such that $f_n \to f$ a.e., and suppose $\int f_n \to \int f < \infty$. Show that for each measurable set $E$ we have $\int_E f_n \to \int_E f$.

Apply the Generalised Lebesgue Th.

$|f_n - f| \le |f_n| + |f| = f_n + f$

$\downarrow$

$f + f$

2. (3rd: P.93, Q10)

   a. Show that if $f$ is integrable over $E$, then so is $|f|$ and

   $$\left|\int_E f\right| \le \int_E |f|.$$

   Does the integrability of $|f|$ imply that of $f$? (no as $f$ is not nece. measurable)

   b. The improper Riemann integral of a function may exist without the function being integrable (in the sense of Lebesgue), e.g., if $f(x) = \frac{\sin x}{x}$ on $[0, \infty]$. If $f$ is integrable, show that the improper Riemann integral is equal to the Lebesgue integral when the former exists.

$\frac{-d\cos x}{x} \to \cos x \cdot \frac{d}{dx}\left(\frac{-1}{x}\right)$

$\int_0^7 + \int_7^\infty$ 1st. integral $\in \mathbb{R}$ by cts. 2nd integral $\in \mathbb{R}$ by inte. by parts

3. (3rd: P.93, Q11)
   If $\varphi$ is a simple function, we have two definitions for $\int \varphi$, that on page 77 and that on page 90 (3rd. ed.). Show that they are the same. (Note: one definition is the one defining at the first stage, the another one is defined by general Lebesgue integral)

Easy from Def.

4. (3rd: P.93, Q12; 4th: P.89, Q30)
   Let $g$ be an integrable function on a set $E$ and suppose that $\{f_n\}$ is a sequence of measurable functions such that $|f_n(x)| \le g(x)$ a.e. on $E$. Show that

   $$\int_E \liminf f_n \le \liminf \int_E f_n \le \limsup \int_E f_n \le \int_E \limsup f_n.$$

Apply Fatou's

All are sem integrable & so finite-valued a.e.

Then, look at $0 \le g \pm f_n$

5. (3rd: P.93, Q13)
   Let $h$ be an integrable function and $\{f_n\}$ a sequence of measurable functions with $f_n \ge -h$ and $\lim f_n = f$. Show that $\int f_n$ and $\int f$ has a meaning and $\int f \le \liminf \int f_n$.

see the next page

$f_n + h$ well-defined $\therefore h \in \mathbb{R}$ a.e.

6. (3rd: P.93, Q14; 4th: P.90, Q33 for part b.)

   a. Show that under the hypotheses of Theorem 17 (3rd. ed.) (i.e. $g_n$, $g$ are integrable such that $g_n \to g$ pointwisely a.e., $f_n$ are measurable, $|f_n| \le g_n$, $f_n \to f$ pointwisely a.e. and $\int g = \lim \int g_n$) we have $\int |f_n - f| \to 0$.

All are integrable & finite-valued a.e

$|f_n - f| \le g_n + g$

$\Downarrow$ $f_n \in \mathcal{L}(E)$ $\because g_n \in \mathcal{L}(E)$

$\Downarrow$ $|f| \le g$ a.e. & $f \in \mathcal{R}(E)$

   b. Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ a.e. with $f$ integrable. Then $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$.

$\downarrow$

$2g$ etc.

Q6(b)

$\Rightarrow$

$$\left|\int |f_n| - \int |f|\right| \overset{\text{linearity}}{=} \left|\int (|f_n| - |f|)\right|$$

$$\leq \int |f_n - f| \quad (\text{Monotone})$$

$$\downarrow 0$$

$\Leftarrow$ All integrable & finite-valued a.e.

so $|f_n - f| \to 0$ as $f_n \to f$ a.e.

From

$$|f_n - f| \leq |f_n| + |f| \overset{\text{def}}{=\!=} g_n$$

$$\downarrow \qquad\qquad \downarrow$$

$$2|f| \overset{\text{def}}{=\!=} g$$

$\int g_n \to \int g$ by assumption.

so the Generalized Lebesgue Th applies

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Q 15 of § 4.4 (Density) Let $f \in \mathcal{L}(\mathbb{R})$, $\varepsilon > 0$.

Progressitively show that $\exists$

$\varphi, \psi, g$ vanishing outside some

with simple $\varphi$, step-function $\psi$ and cts $g$

such that under the $L_1$-norm $\left(\|f\| := \int_{\mathbb{R}} |f|\right)$

$\|f - \varphi\|, \|f - \psi\|, \|f - g\| < \varepsilon$ : wlog $f \geq 0$, $\|f\| > 0$.

$0 \le \varphi_n := n\chi_{A_n} + \sum_{k=1}^{n\cdot 2^n} k\chi_{B_{n,k}} \uparrow f$

(so $0 \le f - \varphi_n \le f$ & apply Lebesgue Conv. Th), $\|\varphi_n - f\| < \varepsilon$.

where $A_n := \{x \in [-n, n] : f(x) \geq n\}$

$B_{n,k} := \{x \in [-n, n] : \frac{k-1}{2^n} \le f(x) < \frac{k}{2^n}\}$

$(k = 1, 2, \dots, n\cdot 2^n)$

Writing in canonical form (with large enough $n$)

$\varphi := \varphi_n := \sum_{j=1}^{N} b_j \chi_{B_j}$ (each $B_j \subseteq [-n, n]$)

Take $U_j$ (representable as a disjoint finitely many intervals) s.t.

$$m(B_j \triangle U_j) < \frac{\varepsilon/N}{(2M)}$$

where $M = \sum_{j=1}^{N} |b_j|$. Then

$$\int |\varphi - \psi| \le \sum_{j=1}^{N} \int_{B_j \triangle U_j} |\varphi - \psi| < 2M \cdot \frac{\varepsilon/N}{(2M)} N = \frac{\varepsilon}{N} \cdot N = \varepsilon.$$

Since each $B_j \subseteq [-n, n]$, $\exists$ a finite-length interval $(a, b) \supseteq B_j, U_j$ $\forall j = 1, 2, \dots, N$.

(so, $\varphi$ and $\psi$ vanishes outside $(a, b)$).

By smoothing the jumps of the steps of $\psi$ (how?), $\exists$ cts $g$ vanishing outside $(a,b)$ s.t. $\|\psi - g\| < \varepsilon$. Thus

$$\|f-\varphi\|,\ \|f-\psi\|,\ \|f-g\| < 3\varepsilon.$$

Q16. By Q15, $\forall \varepsilon>0$, $\exists \psi \in St$ vanishing outside some $[a,b] \subseteq \mathbb{R}$ s.t. $\|f-\psi\| < \varepsilon$.

Since

$$\int_{-\infty}^{\infty} \cos nx\, dx = \frac{\sin nx}{n}\Big|_a^b \to 0 \text{ as } n \to \infty$$

one has

$$\int_{-\infty}^{\infty} \psi(x) \cos nx\, dx \to 0 \quad \text{as } n \to \infty.$$

so $\exists N \in \mathbb{N}$ s.t $\left|\int_{-\infty}^{\infty} \psi(x)\cos nx\, dx\right| < \varepsilon$ $\forall n \geq N$

and so

$$\left|\int_{-\infty}^{\infty} f(x)\cos nx\right| = \left|\int_{-\infty}^{\infty} [f(x)-\psi(x)+\psi(x)] \cos nx\, dx\right|$$

$$\leq \int_{-\infty}^{\infty} |f(x)-\psi(x)|\,|\cos nx|\, dx + \left|\int \psi(x)\cos nx\, dx\right|$$

$$< \varepsilon + \varepsilon = 2\varepsilon, \quad \forall n \geq N.$$

Q 17(b). Only need to show that

$$\lim_{t\to 0} \int_{-\infty}^{\infty} (f(x) - f(x+t))dx = 0 \quad \forall f \in \mathcal{L}(\mathbb{R}).$$

For this, let $\varepsilon > 0$. Take $g$ cts vanishing outside $[a,b]$ as in Q15 (so $\|f-g\| < \varepsilon$). Then

$g$ is uniformly cts on $\mathbb{R}$ (why?), so $\exists\, \delta \in (0,1)$.

$$|g(x) - g(x+t)| < \varepsilon / 2+(b-a) \quad \forall x \in \mathbb{R}, \forall |t| \leq \delta$$

Then

$$\left| \int_{-\infty}^{\infty} (f(x) - f(x+t))dx \right|$$

$$= \left| \int_{-\infty}^{\infty} (f(x) - g(x) + g(x) - g(x+t) + g(x+t) - f(x+t))dx \right|$$

$$\overset{\text{(part a)}}{\leq} \left| \int_{-\infty}^{\infty} (f(x) - g(x))dx \right| + \left| \int_{-\infty}^{\infty} (g(y) - f(y))dy \right| + \int_{-\infty}^{\infty} |g(x) - g(x+t)|dx$$

$$< \varepsilon + \varepsilon + \frac{\varepsilon}{2+(b-a)} \cdot ((b-a)+2) \qquad \left( \begin{array}{l} g(x) - g(x+t) = 0 \\ \quad \forall x \notin [a-1, b+1] \\ |t| < \delta < 1 \end{array} \right)$$

$$= 3\varepsilon.$$

Q18. Let $t_0 \in [0,1]$, $(t_n) \subseteq [0,1] \setminus t_0$ convergent to $t_0$. Suffices to show that

$$\lim_n \int_0^1 f(x, t_n)\, dx = \int_0^1 \lim_n f(x, t_n)\, dx$$

$\lim_n f(x,t_n) \to f(x)$ by given.

Letting $f_n(x) = f(x, t_n)\ \forall x$

apply the Lebesgue Conv. Th.

Q19. Let $t_0 \in [0,1]$. Wish to show

$$\lim_n \frac{\int_0^1 f(x,t_n) - \int_0^1 f(x,t_0)\, dx}{t_n - t_0} = \int_0^1 \left( \lim_n \frac{f(x,t_n) - f(x)}{t_n - t_0} \right) dx$$

$\to$ by given $\dfrac{\partial f(x,t_0)}{\partial t}$

Apply the Mean-Value Th (e.g. 2060),

& note that $\exists\ \bar{t}_n$ lying between $t_n$ & $t_0$ s.t.

$$\left| \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0} \right| = \partial f(x, \bar{t}_n) \leq g(x) \text{ a.e.}$$

so apply the Bounded Conv. Th. Done for Q19.

Q13. The subtle part of the question is:

$$\mathcal{MF}(\mathbb{R}) \ni f := g + h$$

$$g \in \mathcal{MF}^+(\mathbb{R}) \quad \left(\text{so } \int g \in [0,\infty]\right)$$

$$h \in \mathcal{L}(\mathbb{R})$$

Need to justify the definition that

$$\int f := \int g + \int h$$

$$\stackrel{?}{=} \int \bar{g} + \int \bar{h} \quad \text{if also } f = \bar{g} + \bar{h}$$

$$\text{with } \bar{g} \in \mathcal{MF}^+(\mathbb{R}),\ \bar{h} \in \mathcal{L}(\mathbb{R}).$$

Show that it cannot happen that one and only one of $\int g, \int \bar{g}$ is finite (the other is infinite), say

$$\int g \in \mathbb{R}$$

$$\int \bar{g} = +\infty.$$

Note that $f = g + h = \bar{g} + \bar{h}$ and that $g, h, \bar{h}$ are of finite-valued a.e and so

$$\bar{g} = g + h - \bar{h}$$

and

$+\infty = \int \bar{g} = \int (g + h - \bar{h}) = \int g + \int h - \int \bar{h} \in \mathbb{R},$

which is not possible. Therefore

$$\int f := \int g + \int h$$

is well-defined regardless

$\int g$ is finite or infinite.