Hw 6

Q1 (cf. page 72 of Roydon, 3rd Ed but cannot find in 4th Ed). Assume the following version of Non-measurable sets: $\exists\ P_0, P_1, P_2 \cdots \subseteq [0,1)$ s.t.

$P_0 = P, \quad m^*(P_0) = m^*(P_i)\ \forall\ i$

with $[0,1) = \displaystyle\bigcup_{i=1}^{\infty}\!{}_{\circ}\, P_i$. We note that

$$[0,1) = \bigcup_{i=}^{\infty}\!{}_{\circ} \left[\frac{1}{2^{i+1}}, \frac{1}{2^i}\right) \cup \{0\}$$

and define $f \equiv 0$ on $[0,1)$

$f_t$ $(t \in (0,1))$: from $[0,1)$ to $\{0,1\}$

by

$$f_t(x) = \begin{cases} 1 & \text{if } x \in P_n \text{ and } x = 2^{i+1}t - 1 \in [0,1) \\ 0 & \text{otherwise (but } x \in [0,1)) \end{cases}$$

(because, when ~~t~~ $\to 0+$, $2^{i+1}t - 1 \notin [0,1)$ & so $\lim_{t \to 0+} f_t(x) = 0 = f(x)$). Thus the assumptions of non-sequential version Littewood 3rd Principle ($=$ Egoroff Th) are met:

$f_t, f \in \mathcal{MF}^+(E)$ with $E = [0,1)$,

$f_t \to f$.

Q2*(Continued from Q1). Suppose $\varepsilon = 1/2$, $A \subseteq [0,1)$, and $T \in (0, \frac{1}{4})$ s.t

(*) $|f_t(x) - 0| < \frac{1}{2}$ $\forall t \in (0,T)$, $\forall x \in [0,1) \setminus A$.

Show that $A$ cannot be of arbitrary outer measure: $m^*(A) \geq m^*(P)$:

Infact, if $i \in \mathbb{N}$ and $\frac{1}{2^i} < T$, then

$(**) \quad P_i \subseteq A$ .

Q3*. Let $f: [a,b] \to [m, M] \ (\subseteq \mathbb{R})$, $\delta > 0$

and

$$f_\delta(x) := \inf\{f(y) : y \in [a,b] \cap V_\delta(x)\}$$
$$f^\delta(x) := \sup\{ - - - - —- \cdot\},$$
$$\underline{f}(x) := \sup_{\delta > 0} f_\delta(x)$$
$$\bar{f}(x) := \inf_{\delta > 0} f^\delta(x)$$

Show that

$\underline{f}(x) = \bar{f}(x)$ iff $f$ is cts at $x$

Q4 : Let $f : [a,b] \to [m, M] \ (\subseteq \mathbb{R})$, $n \in \mathbb{N}$.
and let $P_n \in \mathcal{P}[a,b]$ obtained by dividing $[a,b]$ into $2^n$-many subintervals of equal length $\left(= \frac{b-a}{2^n}\right)$:

$$P_n = \left\{ I_k^{(n)} : k = 1, 2, \cdots 2^n \right\}$$

where

$$I_k^{(n)} = \left[ a + \frac{(k-1)(b-a)}{2^n}, \ a + \frac{k(b-a)}{2^n} \right] = \left[ a_k^{(n)}, b_k^{(n)} \right]$$

$\forall \ n, k$.

Set, $\forall \ n, k$,

$$m_k^{(n)} = \inf\{ f(y) : y \in \mathrm{int}(I_k^{(n)}) \}$$

$$M_k^{(n)} = \sup\{ \cdots\cdots\cdots \}$$

$$\varphi_n := \sum_{k=1}^{2^n} m_k^{(n)} \chi_{\mathrm{int}(I_k^{(n)})}, \quad \psi_n := \sum M_k^{(n)} \chi_{\cdots}$$

$A_n$ = the end-points of $I_k^{(n)}$, $k = 1, 2, \cdots, 2^n$.

(so $\varphi_n \leq f \leq \psi_n$ on $[a,b] \setminus A_n$)

$\varphi_n, \psi_n \in St[a,b] \subseteq \mathcal{S}[a,b]$

$(R)\int_a^b \varphi_n$ = a lower-Riemann Sum $= (\mathcal{L})\int_{[a,b]} \varphi_n$

$(R)\int_a^b \psi_n$ = a upper- ------- $= (\mathcal{L})\int_{[a,b]} \psi_n$

Q5*.

Show that (with $A := \bigcup_{n\in\mathbb{N}} A_n$), $\forall x \in [a,b]\setminus A$,

(i) $(R)\underline{\int_a^b} f = \lim_n \int_a^b \varphi_n$

(ii) $\forall n \in \mathbb{N}$, $\exists \delta > 0$ s.t. $\varphi_n(x) \le f_\delta(x)$

$\forall \delta > 0$, $\exists n \in \mathbb{N}$ s.t. $\varphi_n(x) \ge f_\delta(x)$

and $\sup_n \varphi_n(x) = \sup_{\delta > 0} f_\delta(x) = \underline{f}(x)$.

Similarly, show that $\lim_n \psi_n(x) = \bar{f}(x)$

(iii) $(R)\underline{\int_a^b} f = (\mathcal{L})\int_{[a,b]} \underline{f}$ and $(R)\overline{\int_a^b} = (\mathcal{L})\int_{[a,b]} \bar{f}$

(iv) $f \in \mathcal{R}[a,b]$ iff $(\mathcal{L})\int_{[a,b]} \bar{f} = (\mathcal{L})\int_{[a,b]} \underline{f}$ iff $\bar{f} = \underline{f}$ a.e. on $[a,b]$

(v) $f \in \mathcal{R}[a,b]$ iff $f$ is cts at $x$ for a.e. $x$ in $[a,b]$.