HW 4_2023 (Ans all questions)

1. Show the following "quasi-regularity" properties for outer-measure m^*: Let $m^*(A) < +\infty$. Then
   (i) $m^*(A) = \inf\{m(G) : \text{open } G \supseteq A\}$
   (ii) $\exists$ a G_δ-set $H := \bigcap_{n \in \mathbb{N}} G_n \supseteq A$ s.t. $m(H) = m^*(A)$ (where each G_n is open).

2. Let $\{E_n : n \in \mathbb{N}\}$ be a sequence of measurable sets and let $E = \liminf E_n \left(:= \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} E_k = \bigcup_{n=1}^{\infty} T_n \text{ where } T_n := \bigcap_{k \geq n} E_k \ \forall n \right)$. Show that

$$m(E) \leq \liminf_n m(E_n)$$

via the following consideration

$$m(E) = \lim_n m(T_n) = \liminf_n m(T_n) \leq \liminf_n m(E_n).$$

3. Let $I := [a,b]$ be a nonempty finite-length interval. Show that it intersects its "shifted interval" $z + I$ and that $I \cup (z+I)$ is an interval with length $\leq (1+\delta) \cdot \ell(I)$ provided that $|z| < \delta \cdot \ell(I)$ and $\delta \in (0,1)$.

4. Let E be a measurable set with non-zero finite measure. Show that $E - E$ contains 0 as an interior point:

Let $\alpha \in (\frac{3}{4}, 1)$. Then $\exists$ an open interval I_α such that

① $\alpha \cdot \ell(I_\alpha) < m(E \cap I_\alpha)$

and

② $V_{\frac{\ell(I_\alpha)}{2}}(0) \subseteq (E \cap I_\alpha) - (E \cap I_\alpha)$

(w.l.g. $m(E) \neq \infty$). $\quad (\subseteq E - E)$

Indeed, for ①, take a COIC $\{I_n : n \in \mathbb{N}\}$ of E s.t.

$$m\Big(\bigcup_{n=1}^{\infty}{}_{\circ} I_n\Big) \geq m(E) > \alpha \cdot \sum_{n=1}^{\infty} \ell(I_n)$$

$$\Vert$$

$$\sum_{n=1}^{\infty} m(E \cap I_n) \quad \text{and hence}$$

$\exists$ at least one term $m(E \cap I_n) > \alpha\, \ell(I_n)$ for some $n \in \mathbb{N}$.

To show ②, suppose otherwise that $\exists\, z \in LHS$ $\left(|z| < \frac{\ell(I_\alpha)}{2}\right)$ s.t. $z \notin RHS$:

$z + (E \cap I_\alpha)$ disjoint from $(E \cap I_\alpha)$ $\delta = 1/2$

and it follows from Q3 (applied to I_α for I)

and $\Big(z + (E \cap I_\alpha)\Big) \cup_{\circ} \Big(E \cap I_\alpha\Big) \subseteq (z + I_\alpha) \cup (I_\alpha)$

of mea $< \frac{3}{2}\, \ell(I_\alpha)$

that

$$2 \cdot m(E \cap I_\alpha) = m\Big(z + (E \cap I_\alpha)\Big) + m\Big(E \cap I_\alpha\Big) < \frac{3}{2}\, \ell(I_\alpha)$$

$$\geq$$

$2 \cdot [\alpha \cdot \ell(I_\alpha)]$, leading to $2\alpha < \frac{3}{2}$, contradicting $\alpha \in (\frac{3}{4}, 1)$

Note. The result in Q4 is known as the Steinhaus Theorem. Another proof is given below.

5. Let $0 < m(E)$ (w.l.g. E is bounded).
By the inner- and outer regularity applied to small enough $\varepsilon > 0$, $\exists$ open G and closed (so compact by the Heine-Borel Th) K with $K \subseteq E \subseteq G$ such that $2\,m(K) > m(G)$.
Then, the standard compactness implies $\exists\, \delta > 0$ s.t. $K + V_\delta(0) \subseteq G$. It remains to show

(*) $V_\delta(0) \subseteq K - K \ (\subseteq E - E)$.

If not, $\exists\, z \in$ LHS but $z \notin$ RHS then $z + K$ and K are disjoint (subsets of G as $z + K \subseteq V_\delta(0) + K \subseteq G$)

$$m\Big((z+K) \cup_0 K\Big) \leqslant m(G)$$
$$\|$$
$$2\,m(K),$$

contradicting our choice of K, G.

7. Let $K \subseteq G \subseteq \mathbb{R}$ with compact K and open G. Then $\exists$ open set V containing 0 (the zero) such that $K + V \subseteq G$.

Hint. $\forall k \in K$, $\exists\, \delta_k > 0$ s.t. $k + V_{2\delta_k}(0) \subseteq G$. By Q6

$$\exists\, k_1, k_2, \dots k_n \in K \text{ s.t. } K \subseteq \bigcup_{i=1}^{n} \left(k_i + V_{\delta_{k_i}}(0)\right).$$

Let $\delta = \min\{\delta_{k_1}, \delta_{k_2}, \dots \delta_{k_n}\}$. Then $K + V_\delta(0) \subseteq G$.

8*. (2nd proof of Steinhaus Th, cf Q5). Given $0 < m(E)$, we assume w.l.g (why?) that E is bounded. You can(?) use the outer & inner regularity with suitably small $\varepsilon > 0$ to find closed K and open G such that $K \subseteq E \subseteq G$ with $2 \cdot m(K) > m(G)$. By Q6, 7, K is compact and $K + V \subseteq G$ for some open set containing 0. Show that $V \subset K - K \; (\subseteq E - E)$, showing Steinhaus Th.

Hint: Let $v \in V$. Should $v + K$ be disjoint from K, one would

$$2\, m(K) = m(v+K) + m(K) = m\left((v+K) \cup_{\circ} K\right) \leq m(G),$$

contradicting our choice of K, G. Therefore $(v+K) \cap K$ is non-empty so $\exists\, k_1, k_2 \in K$ such that $v + k_1 = k_2$; hence $v \in K - K$.