1. Covariant derivatives (for your reference)

Let $\mathbf{V}$ be a vector field on a regular surface M. Let $p \in M$ and $v \in T_p(M)$. Consider the curve α with $\alpha(0) = p$, $\alpha'(0) = v$, then the *covariant derivative* of $\mathbf{V}$ in the direction v is defined as

$$\nabla_v \mathbf{V} = \left(\frac{d}{dt}\mathbf{V}\right)^T \tag{1}$$

where $\mathbf{W}^T$ denotes the tangential part of $\mathbf{W}$.

$\nabla_v \mathbf{V}$ **in local coordinates**: Let $\mathbf{X}(u_1, u_2)$ be local parametrization. Then

$$\mathbf{V} = b_i \mathbf{X}_i$$

where b_i are functions of (u_1, u_2). Suppose $v = v_i \mathbf{X}_i$ and along $\alpha(t)$, $b_i = b_i(u_1(t), u_2(t)) =: b_i(t)$, then

$$\begin{aligned}
\frac{d}{dt}\mathbf{V} =& \frac{d}{dt}\left(b_i(u_1(t), u_2(t))\mathbf{X}_i(u_1(t), u_2(t))\right)\\
=& b_i'(t)\mathbf{X}_i(u_1(t), u_2(t)) + b_i(t)\frac{d}{dt}\mathbf{X}_i(u_1(t), u_2(t))\\
=& b_i'(t)\mathbf{X}_i(u_1(t), u_2(t)) + b_i(t)u_j'(t)\mathbf{X}_{ij}
\end{aligned}$$

Hence (as $\alpha'(0) = u_i'\mathbf{X}_i = v_i\mathbf{X}_i$, so that $u_i'(0) = v_i$):

$$\begin{aligned}
\nabla_v \mathbf{V} =& b_i'(0)\mathbf{X}_i(p) + b_i(0)u_j'(0)\Gamma_{ij}^k(p)\mathbf{X}_k(p)\\
=& b_k'(0)\mathbf{X}_k(p) + b_i(0)v_j\Gamma_{ij}^k(p)\mathbf{X}_k(p)\\
=& \left(b_k' + \Gamma_{ij}^k b_i v_j\right)\mathbf{X}_k.
\end{aligned} \tag{2}$$

So $\nabla_v \mathbf{V}$ depends only on v and the value of $\mathbf{V}$ along the curve α.

2. Covariant derivatives and Gaussian curvature (for your reference)

In particular, if $\mathbf{V} = \mathbf{X}_i$, and $v = \mathbf{X}_j$, then

$$\nabla_{\mathbf{X}_i}\mathbf{X}_j = \Gamma_{ij}^k \mathbf{X}_k. \tag{3}$$

So

$$\begin{aligned}
\nabla_{\mathbf{X}_1}\left(\nabla_{\mathbf{X}_2}\mathbf{X}_1\right) =& \nabla_{\mathbf{X}_1}\left(\Gamma_{21}^k\mathbf{X}_k\right)\\
=& \Gamma_{21,1}^k\mathbf{X}_k + \Gamma_{21}^k\Gamma_{1k}^l\mathbf{X}_l.
\end{aligned}$$

Similarly,

$$\nabla_{\mathbf{X}_2}\left(\nabla_{\mathbf{X}_1}\mathbf{X}_1\right) = \Gamma_{11,2}^k\mathbf{X}_k + \Gamma_{11}^k\Gamma_{2k}^l\mathbf{X}_l.$$

$$\begin{aligned}\langle(\nabla_{\mathbf{X}_1}\nabla_{\mathbf{X}_2}-\nabla_{\mathbf{X}_2}\nabla_{\mathbf{X}_1})\,\mathbf{X}_1,\mathbf{X}_2\rangle &= \left(\Gamma^k_{21,1}-\Gamma^k_{11,2}\right)g_{k2}+\left(\Gamma^k_{21}\Gamma^l_{1k}-\Gamma^k_{11}\Gamma^l_{2k}\right)g_{l2}\\ &= g_{k2}\left(\Gamma^k_{21,1}-\Gamma^k_{11,2}+\Gamma^l_{21}\Gamma^k_{1l}-\Gamma^l_{11}\Gamma^k_{2l}\right)\\ &= g_{12}\left(\Gamma^1_{21,1}-\Gamma^1_{11,2}+\Gamma^l_{21}\Gamma^1_{1l}-\Gamma^l_{11}\Gamma^1_{2l}\right)\\ &\quad+g_{22}\left(\Gamma^2_{21,1}-\Gamma^2_{11,2}+\Gamma^l_{21}\Gamma^2_{1l}-\Gamma^l_{11}\Gamma^2_{2l}\right)\end{aligned}$$

Similarly

$$\begin{aligned}\langle(\nabla_{\mathbf{X}_2}\nabla_{\mathbf{X}_1}-\nabla_{\mathbf{X}_1}\nabla_{\mathbf{X}_2})\,\mathbf{X}_2,\mathbf{X}_1\rangle &= g_{21}\left(\Gamma^2_{12,2}-\Gamma^2_{22,1}+\Gamma^l_{12}\Gamma^2_{2l}-\Gamma^l_{22}\Gamma^2_{1l}\right)\\ &\quad+g_{11}\left(\Gamma^1_{12,2}-\Gamma^1_{22,1}+\Gamma^l_{12}\Gamma^1_{2l}-\Gamma^l_{22}\Gamma^1_{1l}\right)\end{aligned}$$

Now

$$\begin{aligned}2K &= g^{ij}\left(\Gamma^k_{ij,k}-\Gamma^k_{ik,j}+\Gamma^k_{lk}\Gamma^l_{ji}-\Gamma^k_{lj}\Gamma^l_{ki}\right)\\ &= g^{11}\left(\Gamma^k_{11,k}-\Gamma^k_{1k,1}+\Gamma^k_{lk}\Gamma^l_{11}-\Gamma^k_{l1}\Gamma^l_{k1}\right)\end{aligned}$$

$$\begin{aligned}g^{11}\left(\Gamma^k_{11,k}-\Gamma^k_{1k,1}+\Gamma^k_{lk}\Gamma^l_{11}-\Gamma^k_{l1}\Gamma^l_{k1}\right) &= g^{11}\left(\Gamma^2_{11,2}-\Gamma^2_{12,1}+\Gamma^2_{l2}\Gamma^l_{11}-\Gamma^2_{l1}\Gamma^l_{21}\right)\\ &= (\det(g))^{-1}g_{22}\left(\Gamma^2_{11,2}-\Gamma^2_{12,1}+\Gamma^2_{l2}\Gamma^l_{11}-\Gamma^2_{l1}\Gamma^l_{21}\right)\\ g^{22}\left(\Gamma^k_{22,k}-\Gamma^k_{2k,2}+\Gamma^k_{lk}\Gamma^l_{22}-\Gamma^k_{l2}\Gamma^l_{k2}\right) &= g^{22}\left(\Gamma^1_{22,1}-\Gamma^1_{21,2}+\Gamma^1_{l1}\Gamma^l_{22}-\Gamma^1_{l2}\Gamma^l_{12}\right)\\ &= (\det(g))^{-1}g_{11}\left(\Gamma^1_{22,1}-\Gamma^1_{21,2}+\Gamma^1_{l1}\Gamma^l_{22}-\Gamma^1_{l2}\Gamma^l_{12}\right)\\ g^{12}\left(\Gamma^k_{12,k}-\Gamma^k_{1k,2}+\Gamma^k_{lk}\Gamma^l_{21}-\Gamma^k_{l2}\Gamma^l_{k1}\right) &= g^{12}\left(\Gamma^1_{12,1}-\Gamma^1_{11,2}+\Gamma^1_{l1}\Gamma^l_{21}-\Gamma^1_{l2}\Gamma^l_{11}\right)\\ &= -(\det(g))^{-1}g_{12}\left(\Gamma^1_{12,1}-\Gamma^1_{11,2}+\Gamma^1_{l1}\Gamma^l_{21}-\Gamma^1_{l2}\Gamma^l_{11}\right)\\ g^{21}\left(\Gamma^k_{21,k}-\Gamma^k_{2k,1}+\Gamma^k_{lk}\Gamma^l_{12}-\Gamma^k_{l1}\Gamma^l_{k2}\right) &= g^{21}\left(\Gamma^2_{21,2}-\Gamma^2_{22,1}+\Gamma^2_{l2}\Gamma^l_{12}-\Gamma^2_{l1}\Gamma^l_{22}\right)\\ &= -(\det(g))^{-1}g_{21}\left(\Gamma^2_{21,2}-\Gamma^2_{22,1}+\Gamma^2_{l2}\Gamma^l_{12}-\Gamma^2_{l1}\Gamma^l_{22}\right).\end{aligned}$$

Hence

(4)

$$-2K = (\det(g))^{-1}\left[\langle(\nabla_{\mathbf{X}_1}\nabla_{\mathbf{X}_2}-\nabla_{\mathbf{X}_2}\nabla_{\mathbf{X}_1})\,\mathbf{X}_1,\mathbf{X}_2\rangle+\langle(\nabla_{\mathbf{X}_2}\nabla_{\mathbf{X}_1}-\nabla_{\mathbf{X}_1}\nabla_{\mathbf{X}_2})\,\mathbf{X}_2,\mathbf{X}_1\rangle\right].$$

Now if $\mathbf{V},\mathbf{W}$ are two vector fields on M, then

$$\begin{aligned}\frac{\partial}{\partial u_1}\langle\mathbf{V},\mathbf{W}\rangle &= \langle\frac{\partial}{\partial u_1}\mathbf{V},\mathbf{W}\rangle+\langle\mathbf{V},\frac{\partial}{\partial u_1}\mathbf{W}\rangle\\ &= \langle\nabla_{\mathbf{X}_1}\mathbf{V},\mathbf{W}\rangle+\langle\mathbf{V},\nabla_{\mathbf{X}_1}\mathbf{W}\rangle.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{\partial^2}{\partial u_1\partial u_2}\langle\mathbf{X}_1,\mathbf{X}_2\rangle\\ =&\frac{\partial}{\partial u_1}\left(\langle\nabla_{\mathbf{X}_2}\mathbf{X}_1,\mathbf{X}_2\rangle+\langle\mathbf{X}_1,\nabla_{\mathbf{X}_2}\mathbf{X}_2\rangle\right)\\ =&\langle\nabla_{\mathbf{X}_1}\nabla_{\mathbf{X}_2}\mathbf{X}_1,\mathbf{X}_2\rangle+\langle\nabla_{\mathbf{X}_2}\mathbf{X}_1,\nabla_{\mathbf{X}_1}\mathbf{X}_2\rangle+\langle\nabla_{\mathbf{X}_1}\mathbf{X}_1,\nabla_{\mathbf{X}_2}\nabla_{\mathbf{X}_2}\rangle+\langle\mathbf{X}_1,\nabla_{\mathbf{X}_1}\nabla_{\mathbf{X}_2}\mathbf{X}_2\rangle\end{aligned}$$

$$\frac{\partial^2}{\partial u_2 \partial u_1}\langle \mathbf{X}_1, \mathbf{X}_2\rangle$$
$$= \langle \nabla_{\mathbf{X}_2}\nabla_{\mathbf{X}_1}\mathbf{X}_1, \mathbf{X}_2\rangle + \langle \nabla_{\mathbf{X}_1}\mathbf{X}_1, \nabla_{\mathbf{X}_2}\mathbf{X}_2\rangle + \langle \nabla_{\mathbf{X}_2}\mathbf{X}_1, \nabla_{\mathbf{X}_1}\nabla_{\mathbf{X}_2}\rangle + \langle \mathbf{X}_1, \nabla_{\mathbf{X}_2}\nabla_{\mathbf{X}_1}\mathbf{X}_2\rangle$$

Subtract the two equalities:

$$0 = \langle \left(\nabla_{\mathbf{X}_1}\nabla_{\mathbf{X}_2} - \nabla_{\mathbf{X}_2}\nabla_{\mathbf{X}_1}\right)\mathbf{X}_1, \mathbf{X}_2\rangle - \langle \left(\nabla_{\mathbf{X}_2}\nabla_{\mathbf{X}_1} - \nabla_{\mathbf{X}_1}\nabla_{\mathbf{X}_2}\right)\mathbf{X}_2, \mathbf{X}_1\rangle$$

Conclusion:

$$K = -(\det(g))^{-1}\langle \left(\nabla_{\mathbf{X}_1}\nabla_{\mathbf{X}_2} - \nabla_{\mathbf{X}_2}\nabla_{\mathbf{X}_1}\right)\mathbf{X}_1, \mathbf{X}_2\rangle. \tag{5}$$