Assignment 5, Due 15/11/2022 (Tuesday) on or before 11:59 pm

Please upload your assignment to the Blackboard of this course

(1) Prove that on every compact surface $M \subset \mathbb{R}^3$, there is some point p with $K(p) > 0$.

(2) Let $C = \{(x, y, z) \in \mathbb{R}^3 | \; x^2 + y^2 = 1\}$ be a cylinder. Construct an isometry $F : C \to C$ such that the set of fixed points of F, i.e., the set $\{p \in C | F(p) = p\}$, contains exactly two points. (See Do Carmo, p.232)

(3) (a) Let $\mathbf{X} : U \to M$, $(u_1, u_2) \to \mathbf{X}(u_1, u_2)$, be a coordinate parametization, with U being an open set in $\mathbb{R}^2$. Suppose the first fundamental form in this coordinate satisfies $g_{12} = 0$, and $g_{11} = g_{22} = \exp(2f)$ for some smooth function f, i.e. $g_{ij} = \exp(2f)\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and is zero if $i \neq j$. Show that the Christoffel symbols are

$$\Gamma^k_{ij} = \delta_{ki} f_j + \delta_{kj} f_i - \delta_{ij} f_k$$

where $f_i = \frac{\partial f}{\partial u_i}$ etc.

(b) Show that in this parametrization, the Gaussian curvature is given by:

$$K = -e^{-2f} \Delta f$$

where Δ is the Laplacian operator:

$$\Delta = \frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2}.$$

(4) (a) Consider the metric of the sphere in stereographic projection:

$$g_{ij} = \frac{4}{(1 + |\mathbf{u}|^2)^2} \delta_{ij}$$

where $\mathbf{u} = (u_1, u_2)$. What is the Gaussian curvature?

(b) What is the Gaussian curvature of a surface with first fundamental form given by:

$$g_{ij} = \frac{4}{(1 - |\mathbf{u}|^2)^2} \delta_{ij}$$

(c) Find similar first fundamental form so that the Gaussian curvature is k, where $k \neq 0$ is a constant.

(5) Verify that the surfaces:

$$\mathbf{X}(u, v) = (u \cos v, u \sin v, \log u)$$

and

$$\mathbf{Y}(u,v) = (u\cos v, u\sin v, v)$$

have equal Gaussian curvature at that points $\mathbf{X}(u,v), \mathbf{Y}(u,v)$ but the coefficients of the first fundamental forms at points $\mathbf{X}(u,v), \mathbf{Y}(u,v)$ are not the same.