1) let $\alpha' = \frac{d\alpha}{dt}$, $\dot{\alpha} = \frac{d\alpha}{d\tau}$. Then $\alpha' = \frac{d\alpha}{dt} = \frac{d\alpha}{d\tau}\underbrace{\frac{d\tau}{dt}}_{f'(t)} = f'\dot{\alpha}$

$\tau = f(t)$

Then $\alpha''(t) = f''\dot{\alpha} + f'\frac{d\dot{\alpha}}{dt} = f''\dot{\alpha} + f'(f')\ddot{\alpha} = f''\dot{\alpha} + (f')^2\ddot{\alpha}$.

Since we know $(\alpha'')^T = \lambda\alpha' = \lambda f'\dot{\alpha}$, we have

$$\lambda f'\dot{\alpha} = (\alpha'')^T = (f''\dot{\alpha} + f'^2\ddot{\alpha})^T$$

Then $(\ddot{\alpha})^T = 0$ implies $\lambda f' - f'' = 0$.

$$\Leftrightarrow \lambda = \frac{f''}{f'} \Leftrightarrow \lambda(t) = (\ln f'(t))'$$

Then by FTC we have

$$\ln f'(t) = \int_0^t \lambda(s)ds. \Rightarrow f'(t) = e^{\int_0^t \lambda(s)ds}$$

$$\overset{FTC}{\Rightarrow} f(t) = \int_0^t e^{\int_0^x \lambda(s)ds} dx$$

Then choosing this parameter $\tau = f(t)$, by above $(\ddot{\alpha})^T = 0$ and $\alpha(\tau)$ is a geodesic. //

2) i) For convenience re-label $x = u^1$, $y = u^2$. Then let $\alpha(t) = X(u^1(t), u^2(t))$, $\dot{u}^i = \frac{d}{dt} u^i$

The Lagrangian for the energy is

$$\mathcal{L} = \frac{1}{2}\langle \alpha', \alpha' \rangle = \frac{1}{2}\left( g_{11} (\dot{u}^1)^2 + g_{22} (\dot{u}^2)^2 \right).$$

and the E-L equations are

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial u^1} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}^1}\right) = 0 & (1) \\ \frac{\partial \mathcal{L}}{\partial u^2} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}^2}\right) = 0. & (2) \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial u^1} = \frac{1}{2}\left( g_{11,1} (\dot{u}^1)^2 + g_{22,1} (\dot{u}^2)^2 \right)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{u}^1} = g_{11} \dot{u}^1, \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}^1}\right) = g_{11} \ddot{u}^1 + g_{11,1} (\dot{u}^1)^2 + g_{11,2} \dot{u}^2 \dot{u}^1$$

Using $g_{ij} = \frac{1}{y^2} \delta_{ij}$, (1) becomes

$$-\frac{1}{y^2} \frac{d^2 x}{dt^2} + \frac{2}{y^3} \frac{dx}{dt} \frac{dy}{dt} = 0 \iff \frac{d^2 x}{dt^2} - \frac{2}{y} \frac{dx}{dt} \frac{dy}{dt} = 0.$$

Similarly, we have

$$\frac{\partial \mathcal{L}}{\partial u^2} = \frac{1}{2}\left( g_{11,2} (\dot{u}^1)^2 + g_{22,2} (\dot{u}^2)^2 \right)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{u}^2} = g_{22} \dot{u}^2, \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}^2}\right) = g_{22} \ddot{u}^2 + g_{22,1} \dot{u}^2 \dot{u}^1 + g_{22,2} (\dot{u}^2)^2$$

So (2) becomes

$$-\frac{1}{y^3}\left(\frac{dx}{dt}\right)^2 + \frac{1}{y^3}\left(\frac{dy}{dt}\right)^2 - \frac{1}{y^2}\frac{d^2 y}{dt^2} = 0 \iff \frac{d^2 y}{dt^2} - \frac{1}{y}\left(\frac{dy}{dt}\right)^2 + \frac{1}{y}\left(\frac{dx}{dt}\right)^2 = 0$$

So we have the geodesic equations:

$$\begin{cases} \dfrac{d^2x}{dt^2} - \dfrac{2}{y}\dfrac{dx}{dt}\dfrac{dy}{dt} = 0 \\ \dfrac{d^2y}{dt^2} - \dfrac{1}{y}\left(\dfrac{dy}{dt}\right)^2 + \dfrac{1}{y}\left(\dfrac{dx}{dt}\right)^2 = 0 \end{cases}$$

ii) $\alpha(t) = X(c,t)$ So $\dot{u}^1 = 0$, $\dot{u}^2 = 1$

We have $(\alpha'')^T = (\ddot{u}^k + \Gamma^k_{ij}\dot{u}^i\dot{u}^j)X_k$, $\alpha' = \dot{u}^k X_k$

$\dot{u}^1 = 0 \Rightarrow \ddot{u}^1 = 0$. $\qquad \Rightarrow \alpha' = X_2$.

$\dot{u}^2 = 1 \Rightarrow \ddot{u}^2 = 0$.

when $i=1$, $\Gamma^k_{ij}\dot{u}^i\dot{u}^j = 0$. So only have $i,j=2$ term.

$j=1$, $\Gamma^k_{ij}\dot{u}^i\dot{u}^j = 0$

i.e. $(\alpha'')^T = \left(\Gamma^1_{22}(\dot{u}^2)^2 + \Gamma^2_{22}(\dot{u}^2)^2\right)X_2$

So by comparing with geodesic equations in (i) above, we see that

$\Gamma^1_{22} + \Gamma^2_{22} = \frac{-1}{y}$ and we have

$(\alpha'')^T = -\frac{1}{y}X_2 = -\frac{1}{y}\alpha'$ since $\alpha' = X_2$.

so taking $\lambda = \frac{-1}{y}$ we see that $\alpha$ is a pre-geodesic.

iii) $\alpha(t) = X(R\cos t, R\sin t)$, $R>0$, $0<t<\pi$.

$\alpha' = \dot{u}^k X_k = -R\sin t X_1 + R\cos t X_2$. $\quad \dot{u}^1\dot{u}^2 = -R^2\sin t\cos t$.

$\ddot{u}^1 = -R\cos t$, $\quad \ddot{u}^2 = -R\sin t$, $\quad (\dot{u}^1)^2 = R^2\sin^2 t$.

$(\dot{u}^2)^2 = R^2\cos^2 t$

$$(\alpha'')^T = (\ddot{u}^k + \Gamma^k_{ij}\dot{u}^i\dot{u}^j)X_k$$

$$= (\ddot{u}^1 + \Gamma^1_{11}(\dot{u}^1)^2 + 2\Gamma^1_{12}\dot{u}^1\dot{u}^2 + \Gamma^1_{22}(\dot{u}^2)^2)X_1$$

$$+ (\ddot{u}^2 + \Gamma^2_{11}(\dot{u}^1)^2 + 2\Gamma^2_{12}\dot{u}^1\dot{u}^2 + \Gamma^2_{22}(\dot{u}^2)^2)X_2$$

$$= (-R\cos t + \Gamma^1_{11}R^2\sin^2 t - 2\Gamma^1_{12}R^2\sin t\cos t + \Gamma^1_{22}R^2\cos^2 t)X_1$$

$$+ (-R\sin t + \Gamma^2_{11}R^2\sin^2 t - 2\Gamma^2_{12}R^2\sin t\cos t + \Gamma^2_{22}R^2\cos^2 t)X_2$$

Then comparing with the geodesic equations in (i), we get

$\Gamma^1_{11} = 0$, $\Gamma^1_{22} = 0$, $2\Gamma^1_{12} = \frac{-2}{y} \Rightarrow \Gamma^1_{12} = \frac{-1}{y}$.

$\Gamma^2_{11} = \frac{1}{y}$, $-2\Gamma^2_{12} = 0$, $\Gamma^2_{22} = \frac{-1}{y}$.

Then $(\alpha'')^T = \left(-R\cos t + \frac{2R^2\sin t\cos t}{y}\right)X_1$

$$+ \left(-R\sin t + \frac{R^2\sin^2 t}{y} - \frac{R^2\cos^2 t}{y}\right)X_2$$

$(y = R\sin t)$ $\quad = (-R\cos t + 2R\cos t)X_1$

$$+ \left(-R\sin t + R\sin t - R\frac{\cos^2 t}{\sin t}\right)X_2$$

$$= R\cos t\, X_1 - R\frac{\cos^2 t}{\sin t} X_2 .$$

$$\alpha' = -R\sin t\, X_1 + R\cos t\, X_2 .$$

So choosing $\lambda = -\frac{\cos t}{\sin t}$, we see

$(\alpha'')^T = \lambda \alpha'$. , so $\alpha$ is a pre-geodesic.

3) Torus parametrization

$$X(u,v) = \Big(\underbrace{(a+r\cos v)}_{f(v)}\cos u,\ \underbrace{(a+r\cos v)}_{f(v)}\sin u,\ \underbrace{r\sin v}_{g(v)}\Big)$$

- The geodesic equations for a surface of revolution are

$$\begin{cases} \ddot u + \dfrac{2ff'}{f^2}\dot u \dot v = 0 & (1) \\[2ex] \ddot v - \dfrac{ff'}{(f')^2+(g')^2}\dot u^2 + \dfrac{f'f''+g'g''}{(f')^2+(g')^2}\dot v^2 = 0 & (2) \end{cases}$$

Specializing to our specific $f, g$, we have

$$f' = -r\sin v, \quad f'' = -r\cos v$$

$$g' = r\cos v, \quad g'' = -r\sin v$$

$$(f')^2 + (g')^2 = r^2$$

So (1) becomes $\ddot u + \dfrac{2\cancel{(a+r\cos v)}(-r\sin v)}{(a+r\cos v)^{\cancel{2}}}\dot u \dot v$

$$= \ddot u - \frac{2r\sin v}{(a+r\cos v)}\dot u \dot v = 0 \quad (1^*)$$

(2) becomes

$$\ddot v - \frac{(a+r\cos v)(-r\sin v)}{r^2}\dot u^2 + \frac{(-r\sin v)(-r\cos v) + (r\cos v)(-r\sin v)}{r^2}\dot v^2$$

$$= \ddot v + \frac{ar\sin v + r^2\cos v\sin v}{r^2}\dot u^2 + \frac{\cancelto{0}{r^2\sin v\cos v - r^2\cos v\sin v}}{r^2}\dot v^2$$

$$= \ddot v + \frac{ar\sin v + r^2\cos v\sin v}{r^2}\dot u^2 = 0. \quad (2^*)$$

- geodesic $\alpha$ tangent to parallel $\beta(t) = X(t, \frac{\pi}{2})$, so $\theta = 0$ at intersection $\Rightarrow \cos\theta = 1$.

Clairaut's relation gives

$$\text{const} = (r\cos v(t) + a)\cos\theta(t)$$

$$\underset{\substack{\text{at pt. of} \\ \text{intersection.}}}{=} (r\cos\tfrac{\pi}{2} + a)\cos 0 = a.$$

$$\Rightarrow (r\cos v(t) + a)\cos\theta(t) = a.$$

$$\Rightarrow \cos\theta(t) = \frac{a}{r\cos v(t) + a}. \quad (*)$$

Suppose for contradiction that there is a $t_1$ s.t.

$v(t_1) \in [-\pi, \pi] \setminus [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Then $\cos(v(t_1)) < 0$.

Then above implies

$\cos\theta(t_1) > 1$ which is a contradiction

- Let $\beta(t) = (a\cos t, a\sin t, r)$ be the topmost parallel $(v = \frac{\pi}{2})$.

Reparametrize by arclength:

$$\beta'(t) = (-a\sin t, a\cos t, 0) \Rightarrow \langle \beta'(t), \beta'(t) \rangle = a^2.$$

$$\Rightarrow s = \int_0^t \|\beta'(t)\|\,dt = \int_0^t a\,dt = at \Rightarrow t = \frac{s}{a}.$$

So $\beta(s) = (a\cos\frac{s}{a}, a\sin\frac{s}{a}, r)$

$$\beta'(s) = \left(-\sin\frac{s}{a}, \cos\frac{s}{a}, 0\right)$$

$$\beta''(s) = \left(-\frac{1}{a}\cos\frac{s}{a}, -\frac{1}{a}\sin\frac{s}{a}, 0\right).$$

$$X_u = (-(a+r\cos v)\sin u, (a+r\cos v)\cos u, 0)$$

$$X_v = (-r\sin v\cos u, -r\sin v\sin u, r\cos v)$$

$X_u \times X_v$

$$= \left((ar\cos v + r^2\cos^2 v)\cos u, (ar\cos v + r^2\cos^2 v)\sin u, (ar + r^2\cos v)\sin v\right)$$

$$= \left((ar + r^2\cos v)\cos v\cos u, (ar + r^2\cos v)\cos v\sin u, (ar + r^2\cos v)\sin v\right)$$

$$|X_u \times X_v|^2 = (ar\cos v + r^2\cos^2 v)^2\cos^2 u + (ar\cos v + r^2\cos^2 v)^2\sin^2 v$$

$$+ (ar + r^2\cos v)^2\sin^2 v$$

$$= (ar\cos v + r^2\cos^2 v)^2 + (ar + r^2\cos v)^2\sin^2 v$$

$$= (\cos v(ar + r^2\cos v))^2 + (ar + r^2\cos v)^2\sin^2 v$$

$$= (ar + r^2\cos v)^2.$$

$$\Rightarrow N = (\cos v\cos u, \cos v\sin u, \sin v)$$

$k_g = \langle \beta'(s) \times \beta''(s), N \rangle$, at $v = \frac{\pi}{2}$, $N = (0, 0, 1)$.

So $k_g = \frac{1}{a}\sin^2\frac{s}{a} + \frac{1}{a}\cos^2\frac{s}{a} = \boxed{\frac{1}{a}}$.

4) $\alpha'(s) = e_1(s)\cos\theta(s) + e_2(s)\sin\theta(s)$

$$a := N\times\alpha' = N\times\left(e_1(s)\cos(\theta(s)) + e_2(s)\sin(\theta(s))\right)$$
$$= (N\times e_1(s))\cos(\theta(s)) + (N\times e_2(s))\sin(\theta(s))$$

Note: $(e_1\times e_2)\times e_1 = \cancelto{1}{(e_1\cdot e_1)}\, e_2 - \cancelto{0}{(e_2\cdot e_1)}\, e_1$
$= e_2.$

$(e_1\times e_2)\times e_2 = \cancelto{0}{(e_1\cdot e_2)}\, e_2 - \cancelto{1}{(e_2\cdot e_2)}\, e_1$
$= -e_1.$

So $a = (N\times e_1(s))\cos(\theta(s)) + (N\times e_2(s))\sin(\theta(s))$
$= e_2(s)\cos(\theta(s)) - e_1(s)\sin(\theta(s))$. ✓

$k_g := \langle\alpha'', N\times\alpha'\rangle = \langle\alpha'', a\rangle$
$= -\langle\alpha', a'\rangle$ ✓

$a' = -e_1'(s)\sin(\theta(s)) - e_1(s)\cos(\theta(s))\theta'(s)$
$+ e_2'(s)\cos(\theta(s)) - e_2(s)\sin(\theta(s))\theta'(s)$

So $\langle\alpha', a'\rangle = \cancelto{0}{\langle e_1\cos\theta, -e_1'\sin\theta\rangle} + \langle e_1\cos\theta, -e_1\cos\theta\,\theta'\rangle$
$+ \langle e_1\cos\theta, e_2'\cos\theta\rangle + \cancelto{0}{\langle e_1\cos\theta, -e_2\sin\theta\,\theta'\rangle}$
$+ \langle e_2\sin\theta, -e_1'\sin\theta\rangle + \cancelto{0}{\langle e_2\sin\theta, e_1\cos\theta\,\theta'\rangle}$
$+ \cancelto{0}{\langle e_2\sin\theta, e_2'\cos\theta\rangle} + \langle e_2\sin\theta, -e_2\sin\theta\,\theta'\rangle$

$= -\cos^2\theta\,\theta'(s) + \cos^2\theta\langle e_1, e_2'\rangle - \sin^2\theta\langle e_2, e_1'\rangle - \sin^2\theta\,\theta'(s)$

Note: $\langle e_1, e_2\rangle' = \langle e_1', e_2\rangle + \langle e_1, e_2'\rangle$

$$\Rightarrow \langle e_1, e_2'\rangle = \cancel{\langle e_1, e_2\rangle}^{0}{}' - \langle e_1', e_2\rangle$$

$$= -\langle e_1', e_2\rangle.$$

So $\langle \alpha'', a' \rangle = -\theta'(s) - \left(\cos^2\theta \langle e_1', e_2\rangle + \sin^2\theta \langle e_1', e_2\rangle\right)$

$$= -\theta'(s) - \langle e_1', e_2\rangle.$$

So $k_g = -\langle \alpha'', a'\rangle = \langle e_1', e_2\rangle + \theta'(s)$

$$= \left\langle \left(\frac{X_1}{|X_1|}\right)', \frac{X_2}{|X_2|}\right\rangle + \theta'(s)$$

$$\left( \begin{aligned} |X_1| &= g_{11}^{1/2} = e^f \\ |X_2| &= g_{22}^{1/2} = e^f \end{aligned} \right)$$

$$= \left\langle \left(\frac{X_1}{e^f}\right)', \frac{X_2}{e^f}\right\rangle + \theta'(s)$$

$$= \left\langle \left(\frac{1}{e^f}\frac{d}{ds}(X_1) + X_1\frac{d}{ds}\left(\frac{1}{e^f}\right)\right), \frac{X_2}{e^f}\right\rangle + \theta'$$

$$= e^{-2f}\left\langle \frac{d}{ds}X_1, X_2\right\rangle + \left\langle \frac{d}{ds}\left(\frac{1}{e^f}\right) \cancel{X_1, X_2}^{0} \right\rangle + \theta'$$

$$= e^{-2f}\left\langle \frac{d}{ds}X_1, X_2\right\rangle + \theta'$$

$$= e^{-2f}\langle X_{11}\, u_1' + X_{12} u_2', X_2\rangle + \theta' \quad \left(X_{ij} = \Gamma_{ij}^{k} X_k + h_{ij} N\right)$$

$$= e^{-2f}\langle \cancel{\Gamma_{11}^{1} u_1' X_1} + \Gamma_{11}^{2} u_1' X_2 + \cancel{h_{11} u_1' N}$$

$$+ \cancel{\Gamma_{12}^{1} u_2' X_1} + \Gamma_{12}^{2} u_2' X_2 + \cancel{h_{12} u_2' N}, X_2\rangle + \theta'$$

$$= e^{-2f}\langle \Gamma_{11}^{2} u_1' X_2 + \Gamma_{12}^{2} u_2' X_2, X_2\rangle + \theta'$$

$$= e^{-2f}\left(\Gamma^2_{11} u_1' + \Gamma^2_{12} u_2'\right) g_{22} + \theta' \quad (g_{22} = e^{2f})$$

$$= \left(\Gamma^2_{11} u' + \Gamma^2_{12} v'\right) + \theta' \quad (u_1 = u,\ u_2 = v)$$

$$\Gamma^k_{ij} = \delta_{ki} f_j + \delta_{kj} f_i - \delta_{ij} f_k$$

$$\Rightarrow \Gamma^2_{11} = \delta_{21} f_1 + \delta_{21} f_1 - \delta_{11} f_2 = -\frac{\partial f}{\partial v}.$$

$$\Gamma^2_{12} = \delta_{21} f_2 + \delta_{22} f_1 - \delta_{12} f_2 = \frac{\partial f}{\partial u}$$

So finally, we have

$$k_g = \left(\Gamma^2_{11} u' + \Gamma^2_{12} v'\right) + \theta'$$

$$= \left(-u' \frac{\partial f}{\partial v} + v' \frac{\partial f}{\partial u}\right) + \theta' \checkmark$$