2) Let e_1, e_2 be principal directions with corresponding principal curvatures k_1, k_2. Let λ_j be normal curvatures at $p \in M$ along a direction making angles $\theta_j := \frac{(j-1)2\pi}{m}$ with e_1 (WLOG) for $j = 1, \dots, m$. Then by Euler's formula, we have

$$\sum_{j=1}^{m} \lambda_j = \sum_{j=1}^{m} \left(k_1 \cos^2\theta_j + k_2 \sin^2\theta_j\right) = m k_2 + (k_1 - k_2) \sum_{j=1}^{m} \cos^2\theta_j$$

Now we use the following trigonometric identity:

$$\sum_{j=1}^{m} \cos^2\theta_j = 1 + \cos^2\left(\frac{2\pi}{m}\right) + \cos^2\left(\frac{2 \cdot 2\pi}{m}\right) + \dots + \cos^2\left(\frac{(m-1)2\pi}{m}\right) = \frac{m}{2}:$$

Let $\theta = \frac{2\pi}{m}$, so $\theta_j = (j-1)\theta$, and we have

$$\sum_{j=-(m-1)}^{m-1} e^{2ij\theta} = \sum_{j=-(m-1)}^{m-1} \left(\cos(2j\theta) + i\sin(2j\theta)\right)$$

$$= \left(\sum_{j=0}^{m-1} 2\cos(2j\theta)\right) \qquad \text{since cos is even and sin is odd}$$

$$= \sum_{j=0}^{m-1} \left(4\cos^2(j\theta) - 2\right) = 4\sum_{j=1}^{m} \cos^2(\theta_j) - 2m - 1$$

So $\sum_{j=1}^{m} \cos^2\theta_j = \frac{1}{4}\left(\sum_{j=-(m-1)}^{m-1} e^{2ij\theta} + 2m + 1\right) = \frac{1}{4}(-1 + 2m + 1) = \frac{m}{2}$

by geometric progression.

So the result follows.

3) Proposition 1 of section 3-3 of do Carmo states that at a hyperbolic point $p \in M$, in each neighborhood of p there exist points in both sides of T_pM.

Now suppose $p \in M$ s.t. $K(p) < 0$. Then p is a hyperbolic point, but by the proposition we arrive at a contradiction with the assumption that M lies on one side of the tangent plane. So we conclude $K(p) < 0$ is impossible. //

4) Show that the helicoid

$$X(u,v) = (a\sinh v\cos u,\ a\sinh v\sin u,\ au)$$

and the Enneper's Surface

$$X(u,v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2\right)$$

are minimal surfaces.

Helicoid: one can compute

$$X_u = (-a\sinh v\sin u,\ a\sinh v\cos u,\ a)$$

$$X_v = (a\cosh v\cos u,\ a\cosh v\sin u,\ 0)$$

$$E = \langle X_u, X_u\rangle = a^2\cosh^2 v.$$

$$F = \langle X_u, X_v\rangle = 0$$

$$G = \langle X_v, X_v\rangle = a^2\cosh^2 v.$$

So X is an isothermal parametrization.

$$X_{uu} = (-a\sinh v\cos u,\ -a\sinh v\sin u,\ 0)$$

$$X_{vv} = (a\sinh v\cos u,\ a\sinh v\sin u,\ 0)$$

So $X_{uu} + X_{vv} = 0 \Rightarrow$ minimal.

Enneper's Surface:

$$X(u,v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2\right)$$

$$X_u = (1 - u^2 + v^2,\ 2uv,\ 2u)$$

$$X_v = (2uv,\ 1 - v^2 + u^2,\ -2v)$$

$$E = \langle X_u, X_u \rangle = (1+u^2+v^2)^2$$

$$F = \langle X_u, X_v \rangle = 0.$$

$$G = \langle X_v, X_v \rangle = (1+u^2+v^2)^2$$

So X is an isothermal parametrization

$$X_{uu} = (-2u,\ 2v,\ 2)$$

$$X_{vv} = (2u,\ -2v,\ -2)$$

So $X_{uu} + X_{vv} = 0 \Rightarrow$ minimal.

5) $X(u,v) = (u, h(u)\cos v, h(u)\sin v)$

$X_u = (1, h'(u)\cos v, h'(u)\sin v)$

$X_v = (0, -h(u)\sin v, h(u)\cos v)$

$X_u \times X_v = (h'(u)h(u), -h(u)\cos v, -h(u)\sin v)$

$E = \langle X_u, X_u \rangle = 1 + h'(u)^2$

$F = \langle X_u, X_v \rangle = 0$

$G = \langle X_v, X_v \rangle = h(u)^2$

Then $|X_u \times X_v|^2 = EG - F^2 = h(u)^2 + h'(u)^2 h(u)^2$.

So $N = \dfrac{1}{\sqrt{h(u)^2 + h'(u)^2 h(u)^2}} (h'(u)h(u), -h(u)\cos v, -h(u)\sin v)$

$X_{uu} = (0, h''(u)\cos v, h''(u)\sin v)$

$X_{uv} = (0, -h'(u)\sin v, h'(u)\cos v)$

$X_{vv} = (0, -h(u)\cos v, -h(u)\sin v)$

$e = \langle N, X_{uu} \rangle = \dfrac{-h''}{\sqrt{1+h'^2}}$

$f = \langle N, X_{uv} \rangle = 0$

$g = \langle N, X_{vv} \rangle = \dfrac{h}{\sqrt{1+h'^2}}$

So $H = \frac{1}{2}\frac{eG + gE}{EG}$

$$= \frac{h(u)^2 + h(u)^2 h'(u)^2 - h(u)^3 h''(u)}{2\left(h(u)^2 + h'(u)^2 h(u)^2\right)^{3/2}} = \frac{\left(1 + (h')^2 - h\, h''\right)}{2h\left(1 + h'^2\right)^{3/2}}$$

Suppose $H = \frac{c}{2} \neq 0$, then we get the DE

$$1 + h' - hh'' = ch\left(1 + (h')^2\right)^{3/2}.$$

When $a = \frac{-1}{c} \iff c = \frac{-1}{a}$ and the DE becomes

$$\frac{a(1 + h'^2) - ahh''}{(1 + h'^2)^{3/2}} + h = 0$$

$$\Rightarrow \quad 2h'\left(\frac{a(1 + h'^2) - ahh''}{(1 + h'^2)^{3/2}} + h\right) = 0$$

$$\iff \frac{2ah'(1 + h'^2) - 2ahh'h''}{(1 + h'^2)^{3/2}} + 2hh' = 0$$

$$\Rightarrow \frac{d}{du}\left(\frac{2ah}{(1 + h'^2)^{1/2}} + h^2\right) = 0, \text{ so } h^2 + \frac{2ah}{\sqrt{1 + h'^2}} \equiv \text{const.}$$