1) One can show that for $\alpha(0)=p$, $\alpha'(0)=v$,

$$dF_p(v) = \frac{d}{dt}F(\alpha(t))\Big|_{t=0} = \langle \operatorname{grad} F(\alpha(t)), \alpha'(t)|_{t=0}\rangle$$
$$= \langle \operatorname{grad} F(p), v\rangle$$

where $\langle , \rangle$ denotes the inner product.

which is clearly independent of α, hence well-defined.

Since the inner product is well-defined, we have for $v,w \in T_pM$, $\alpha \in \mathbb{R}$,

$$dF_p(\alpha v + w) = \langle \operatorname{grad} F(p), \alpha v + w\rangle$$
$$= \alpha \langle \operatorname{grad} F(p), v\rangle + \langle \operatorname{grad} F(p), w\rangle$$
$$= \alpha\, dF_p(v) + dF_p(w).$$

So dF_p is linear.

Alternatively, for linearity, have to show

$dF_p(\lambda v) = \lambda\, dF_p(v)$ by taking $\beta(t) := \alpha(\lambda t) = X(u(\lambda t), v(\lambda t))$

so that if $\alpha'(0)=v$, then $\beta'(0)=\lambda v$ and then proceed from definition, and similarly for $v+w$ so that if

$\alpha'(0)=v$, $\beta'(0)=w$, then choose γ so that

$\gamma'(0)=v+w$ and use γ to show

$dF_p(v+w) = dF_p(v) + dF_p(w)$.

2) $\alpha : (0, \frac{\pi}{2}) \to xz$ plane by

$$\alpha(t) = \left(\sin t, 0, \cos t + \log \tan \frac{t}{2}\right).$$

Surface of revolution by rotating α about z-axis is given by

$$X(t, \theta) = \left(\sin t \cos\theta, \sin t \sin\theta, \cos t + \log\tan\frac{t}{2}\right).$$

Then $X_t = \left(\cos t \cos\theta, \cos t \sin\theta, -\sin t + \frac{1}{\sin t}\right)$

$$X_\theta = (-\sin t \sin\theta, \sin t \cos\theta, 0)$$

$$E = \langle X_t, X_t \rangle = \frac{\cos^2 t}{\sin^2 t}$$

$$F = \langle X_t, X_\theta \rangle = 0$$

$$G = \langle X_\theta, X_\theta \rangle = \sin^2 t.$$

$$EG - F^2 = \cos^2 t \Rightarrow |X_t \times X_\theta| = \cos t,$$

$$X_t \times X_\theta = (-\cos^2 t \cos\theta, -\cos^2 t \sin\theta, \sin t \cos t)$$

$$\Rightarrow N = (-\cos t \cos\theta, -\cos t \sin\theta, \sin t)$$

$$X_{tt} = \left(-\sin t \cos\theta, -\sin t \sin\theta, -\cos t - \frac{\cos t}{\sin^2 t}\right)$$

$$X_{t\theta} = (-\cos t \sin\theta, \cos t \cos\theta, 0)$$

$$X_{\theta\theta} = (-\sin t \cos\theta, -\sin t \sin\theta, 0)$$

$e = \langle N, X_{tt} \rangle = \frac{-\cos t}{\sin t}.$

$f = \langle N, X_{t\theta} \rangle = 0.$

$g = \langle N, X_{\theta\theta} \rangle = \sin t \cos t.$

Then $K(p) = \frac{eg - f^2}{EG - F^2} = \frac{-\cos^2 t}{\cos^2 t} = -1$

3) $M = \{(x,y,z) : z = x^2 + ky^2\}$ $k>0$. Can be parameterized by $X(u,v) = (u, v, u^2 + kv^2)$.

Then $X_u = (1, 0, 2u)$
$X_v = (0, 1, 2kv)$.

When $p = (0,0,0)$, $u = v = 0$, and $X_u = e_1$
$X_v = e_2$.

So at $p = (0,0,0)$, e_1, e_2 form a basis of T_pM.

$$N = \frac{1}{\sqrt{1+4u^2+4k^2v^2}}(-2u, -2kv, 1).$$

Note that this satisfies $\langle N, e_3\rangle = \frac{1}{\sqrt{1+4u^2+4k^2v^2}} > 0$.

$E = \langle X_u, X_u\rangle = 1 + 4u^2$ at p, $E|_p = 1$

$F = \langle X_u, X_v\rangle = 4kuv$ $F|_p = 0$

$G = \langle X_v, X_v\rangle = 1 + 4k^2v^2$ $G|_p = 1$

$X_{uu} = (0,0,2)$, $X_{uv} = (0,0,0)$, $X_{vv} = (0,0,2k)$.

So $e = \langle N, X_{uu}\rangle = \frac{2}{\sqrt{1+4u^2+4k^2v^2}}$ at p, $e|_p = 2$

$f = \langle N, X_{uv}\rangle = 0$ at p, $f|_p = 0$.

$g = \langle N, X_{vv}\rangle = \frac{2k}{\sqrt{1+4u^2+4k^2v^2}}$ at p, $g|_p = 2k$

So at p, $S_p = \dfrac{1}{EG-F^2}\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\Bigg|_p$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2k \end{pmatrix}.$$

So the principal curvatures of M at $p=(0,0,0)$ are

$k_1 = 2$, $k_2 = 2k$. $(k>0)$.

4) i) $d(fN)(v) = \langle df(p), v\rangle N(p) + f(p)\, dN_p(v)$ by chain rule and product rule.

Then $\langle d(fN)(v_1) \times d(fN)(v_2), N\rangle$

$$= \det\big(d(fN)(v_1),\, d(fN)(v_2),\, N\big)$$

$$= \det\big(\langle df(p), v_1\rangle N(p) + f(p)\, dN_p(v_1),\ \langle df(p), v_2\rangle N(p) + f(p)\, dN_p(v_2),\ N\big)$$

$$= \det\big(f(p)\, dN_p(v_1),\, f(p)\, dN_p(v_2),\, N\big)$$

$$= f^2 \langle dN_p(v_1) \times dN_p(v_2), N\rangle$$

Let $S_p = [h_{ij}]$ be the matrix of the shape operator at p.

Recall $S_p = -dN_p$, so $dN_p = [-h_{ij}]$ and in the basis $\{v_1, v_2\}$

$$dN_p(v_1) = -h_{11}v_1 - h_{21}v_2, \quad dN_p(v_2) = -h_{12}v_1 - h_{22}v_2.$$

So $dN_p(v_1) \times dN_p(v_2)$

$$= (h_{11}h_{22} - h_{12}h_{21})\, v_1 \times v_2$$ ← since $\{v_1, v_2\}$ is an onb, $N = v_1 \times v_2$.

$$= \det(S_p) N$$

$$= K(p) N.$$

$\Rightarrow \dfrac{\langle d(fN)(v_1) \times d(fN)(v_2), N\rangle}{f^2} = \dfrac{f^2 \langle dN_p(v_1) \times dN_p(v_2), N\rangle}{f^2}$

$= K(p)$ as required //.

vi) $h(x,y,z) := \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

$f(x,y,z) = \left(\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + \dfrac{z^2}{c^4}\right)^{1/2} \Big|_h$.

we can take

$N = \dfrac{dh}{|dh|} = \dfrac{dh}{\left(\left(\frac{2x}{a^2}\right)^2 + \left(\frac{2y}{b^2}\right)^2 + \left(\frac{2z}{c^2}\right)^2\right)^{1/2}} = \dfrac{dh}{\left(4\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\right)^{1/2}} = \dfrac{dh}{2f}$

For a vector $v = (v_1, v_2, v_3)$, we have

$d(fN)(v) = \frac{1}{2} dh(v) = \frac{1}{2}\left(\frac{2x}{a^2}dx + \frac{2y}{b^2}dy + \frac{2z}{c^2}dz\right)(v_1, v_2, v_3)$

$= \left(\dfrac{v_1}{a^2}, \dfrac{v_2}{b^2}, \dfrac{v_3}{c^2}\right)$.

let $\{u, v\}$ be an onbs, $N = u \times v = (n_1, n_2, n_3)$, then by (i), we have

$$K = \frac{1}{f^2}\begin{vmatrix} \frac{u_1}{a^2} & \frac{u_2}{b^2} & \frac{u_3}{c^2} \\ \frac{v_1}{a^2} & \frac{v_2}{b^2} & \frac{v_3}{c^2} \\ n_1 & n_2 & n_3 \end{vmatrix} = \frac{1}{f^2 a^2 b^2 c^2}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ a^2 n_1 & b^2 n_2 & c^2 n_3 \end{vmatrix}$$

$$= \frac{1}{f^2a^2b^2c^2}\langle u \times v, (a^2n_1, b^2n_2, c^2n_3)\rangle$$

$$= \frac{1}{f^2a^2b^2c^2}\langle N, (a^2n_1, b^2n_2, c^2n_3)\rangle$$

$$= \frac{1}{f^2a^2b^2c^2}\langle (a^2, b^2, c^2), (n_1^2, n_2^2, n_3^2)\rangle.$$

From $N = \frac{dh}{2f}$, we get

$$(n_1, n_2, n_3) = \left(\frac{x}{fa^2}, \frac{y}{fb^2}, \frac{z}{fc^2}\right)$$

$$\Rightarrow (n_1^2, n_2^2, n_3^2) = \left(\frac{x^2}{f^2a^4}, \frac{y^2}{f^2b^4}, \frac{z^2}{f^2c^4}\right)$$

So $\langle (a^2, b^2, c^2), (n_1^2, n_2^2, n_3^2)\rangle$

$$= \left\langle (a^2, b^2, c^2), \left(\frac{x^2}{f^2a^4}, \frac{y^2}{f^2b^4}, \frac{z^2}{f^2c^4}\right)\right\rangle$$

$$= \frac{x^2}{f^2a^2} + \frac{y^2}{f^2b^2} + \frac{z^2}{f^2c^2}$$

So $K = \frac{1}{f^2a^2b^2c^2}\left(\frac{x^2}{f^2a^2} + \frac{y^2}{f^2b^2} + \frac{z^2}{f^2c^2}\right) = \frac{1}{f^4a^2b^2c^2}\underbrace{\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)}_{=h=1}$

$$= \frac{1}{f^4a^2b^2c^2} \text{ as required } //.$$

5) Given the parametrization of the catenoid

$$X(u,v) = (a\cosh(v)\cos(u),\ a\cosh(v)\sin(u),\ av) \quad a>0.$$

$$X_u = (-a\cosh(v)\sin(u),\ a\cosh(v)\cos(u),\ 0)$$

$$X_v = (a\sinh(v)\cos(u),\ a\sinh(v)\sin(u),\ a)$$

$$X_u \times X_v = (a^2\cosh(v)\cos(u),\ a^2\cosh(v)\sin(u),\ -a^2\cosh(v)\sinh(v))$$

$$|X_u \times X_v| = a^2\cosh^2(v)$$

So $N = \left(\dfrac{\cos(u)}{\cosh(v)},\ \dfrac{\sin(u)}{\cosh(v)},\ \dfrac{-\sinh(v)}{\cosh(v)}\right)$

$$S_p(X_u) = -N_u = \left(\frac{\sin(u)}{\cosh(v)},\ \frac{-\cos(u)}{\cosh(v)},\ 0\right)$$

$$S_p(X_v) = -N_v = (\cos(u)\tanh(v)\operatorname{sech}(v),\ \sin(u)\tanh(v)\operatorname{sech}(v),\ \operatorname{sech}^2(v))$$

So $E = \langle X_u, X_u\rangle = a^2\cosh^2(v)$

$F = \langle X_u, X_v\rangle = 0$

$G = \langle X_v, X_v\rangle = a^2\cosh^2(v)$.

$e = \langle S_p(X_u), X_u\rangle = -a$

$f = \langle S_p(X_u), X_v\rangle = 0$

$g = \langle S_p(X_v), X_v\rangle = a$.

$$H(p) = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2} = \frac{1}{2}\frac{(-a^3\cosh^2(v) + a^3\cosh^2(v))}{a^4\cosh^4(v)} = 0.$$