Regular surfaces 2: Change of coordinates and smooth structure

Proposition

Let M be a regular surface and let $\mathbf{X} : U \to M$, $\mathbf{Y} : V \to M$ be two coordinate parametrizations. Let $S = \mathbf{X}(U) \cap \mathbf{Y}(V) \subset M$. Let $U_1 = \mathbf{X}^{-1}(S)$ and $V_1 = \mathbf{Y}^{-1}(S)$. Then $\mathbf{Y}^{-1} \circ \mathbf{X} : U_1 \to V_1$ is a diffeomorphism.

Proof

Let $p \in S$. Then there is an open set $S_1 \subset S$ such that S_1 is given by the graph $\{(x, y, z)|(x, y) \in \mathcal{O}, z = f(x, y)\}$. Now if $(u, v) \in U_1$ with $\mathbf{X}(u, v) \in S_1$, then

$$\mathbf{X}(u, v) = (x(u, v), y(u, v), f(x(u, v), y(u, v)))$$

because $z = f(x, y)$.
$\mathbf{X}_u = (x_u, y_u, f_x x_u + f_y y_u), \mathbf{X}_v = (x_v, y_v, f_x x_v + f_y y_v)$. Since $\mathbf{X}_u$ and $\mathbf{X}_v$ are linearly independent, we have $(x_u, y_u), (x_v, y_v)$ are linearly independent (why?). This implies $(u, v) \to (x, y)$ is difffeormphic near $\mathbf{X}^{-1}(p)$. Similarly, if $(\xi, \eta) \in V_1$, then $(\xi, \eta) \to (x, y)$ is diffeomorphic near $\mathbf{Y}^{-1}(p)$. Hence $(\xi, \eta) \to (u, v)$ is diffeomorphic.

Smooth structure

Definition

(i) *Let M be regular surface and let $f : M \to \mathbb{R}$ be a function. f is said to be smooth if and only if $f \circ \mathbf{X}$ is smooth for all coordinate chart $\mathbf{X} : U \to M$.*

(ii) *M_1, M_2 be regular surfaces and let $F : M_1 \to M_2$ be a map. F is said to be smooth if and only if the following is true: For any $p \in M_1$ and any coordinate charts $\mathbf{X}$ of p, $\mathbf{Y}$ of $q = F(p)$, $\mathbf{Y}^{-1} \circ \mathbf{X}$ is smooth whenever it is defined.*

Main point: The concepts are well-defined.

Abstract surfaces: a digression

An abstract surface (**differentiable manifold of dimension two**) is a set M together with a family of one-to-one maps $\mathbf{X}_\alpha : U_\alpha \to M$ of open sets $U_\alpha \subset \mathbb{R}^2$ such that:

$\bigcup_\alpha \mathbf{X}_\alpha(U_\alpha) = M$;

For any α, β, if $W = \mathbf{X}_\alpha(U_\alpha) \cap \mathbf{X}_\beta(U_\beta) \neq \emptyset$, then $V_\alpha = \mathbf{X}_\alpha^{-1}(W)$, $V_\beta = \mathbf{X}_\beta^{-1}(W)$ are open sets in $\mathbb{R}^2$ and $\mathbf{X}_\beta^{-1} \circ \mathbf{X}_\alpha : V_\alpha \to V_\beta$ and $\mathbf{X}_\beta^{-1} \circ \mathbf{X}_\alpha : V_\alpha \to V_\beta$ are diffeomorphisms.