Vector fields along α

We want to discuss variational properties of geodesics. First we want to find a way to construct variations of a curve. The direction of a variation is given by a vector field along the curve. Let $\alpha(t)$ be a regular curve on a regular surface M. A *vector field*

w *along* α is a vector field $\mathbf{w}(t)$ such that:
$\mathbf{w}(t)$ is smooth in t. $\mathbf{w}(t)$ is a vector on the tangent space of M at $\alpha(t)$: $\mathbf{w}(t) \in T_{\alpha(t)}(M)$.

Construction of a variation of α adapted to a given vector field $\mathbf{w}$ along α

Consider a variation $\alpha(s,t)$ with $|s|<\delta$, $t\in[a,b]$ such that Let $\alpha(t)$, $t\in[a,b]$ be a regular curve on M. A *variation of α with end points fixed* is a map

$$\alpha:(-\delta,\delta)\times[a,b]\to M$$

such that: $\alpha(0,t)=\alpha(t)$, the original curve.
$\alpha(s,a)=\alpha(a),\alpha(s,b)=\alpha(b)$, i.e. end points fixed.
$\frac{\partial\alpha}{\partial s}|_{s=0}$ is called the variational vector field. Given a vector field $\mathbf{w}$ along α so that $\mathbf{w}(a)=0,\mathbf{w}(b)=0$, want to find a variation $\alpha(s,t)$ so that $\frac{\partial\alpha}{\partial s}|_{s=0}=\mathbf{w}$.

Construction, cont.

We only consider the case $\mathbf{X}(u^1, u^2)$ is a local parametrization and $\alpha(t) = \mathbf{X}(u^1(t), u^2(t)), t \in [a, b]$.

- Let $\mathbf{w}(t)$ is a vector field along α. Then $\mathbf{w}(t) = \sum_{i=1}^{2} a^i(t)\mathbf{X}_i(u^1(t), u^2(t))$.
- Let $\alpha(s, t) = \mathbf{X}(u^1(t) + sa^1(t), u^2(t) + sa^2(t))$.
- Then $\frac{\partial s}{\partial s}\alpha(s, t)|_{s=0} = \sum_{i=1}^{2} a^i(t)\mathbf{X}_i(u^1(t), u^2(t)) = \mathbf{w}(t)$.

First variation of arc length

Let M be a regular surface. Let $\alpha : [a, b] \to M$ be a regular curve. Then length functional is defined as

$$\ell(\alpha) = \int_a^b |\dot{\alpha}| dt.$$

We want to compute the variation of ℓ around α. Consider a variation $\alpha(s, t)$ with $|s| < \delta$, $t \in [a, b]$ such that

- $\alpha(0, t) = \alpha(t)$, the original curve.
- $\alpha(s, a) = \alpha(a), \alpha(s, b) = \alpha(b)$, i.e. end points fixed.
- Let $\ell(s) = \ell(\alpha_s) = \int_a^b |\dot{\alpha}_s(t)| dt = \int_a^b |\frac{\partial}{\partial t}\alpha(s, t)|$. Here $\alpha_s(t) = \alpha(s, t)$
- Want to compute $\frac{d}{ds}\ell(s)|_{s=0}$.

First variation of arc length, cont.

$$
\begin{aligned}
\frac{d}{ds}\ell(s)|_{s=0} &= \frac{d}{ds}\int_a^b \langle \frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t}\rangle^{\frac{1}{2}} dt \\
&= \int_a^b \langle \frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t}\rangle^{-\frac{1}{2}} \langle \frac{\partial^2\alpha}{\partial s\partial t}, \frac{\partial\alpha}{\partial t}\rangle dt \\
&= \int_a^b \langle \frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t}\rangle^{-\frac{1}{2}} \langle \frac{\partial}{\partial t}\left(\frac{\partial\alpha}{\partial s}\right), \frac{\partial\alpha}{\partial t}\rangle dt
\end{aligned}
$$

At $s = 0$,

- $\frac{\partial\alpha}{\partial t} = \alpha'(t)$, here $\alpha(t) = \alpha(0,t)$ is the original curve.
- $\frac{\partial}{\partial t}\left(\frac{\partial\alpha}{\partial s}\right) = \mathbf{w}'(t)$, where $\mathbf{w} = \frac{\partial\alpha}{\partial s}|_{s=0}$.

First variation of arc length, cont.

$$\frac{d}{ds}\ell(s)|_{s=0} = \int_a^b \langle \mathbf{w}', |\alpha'|^{-1}\frac{d\alpha}{dt}\rangle dt$$
$$= - \int_a^b \langle \mathbf{w}, \frac{d}{dt}\left(|\alpha'|^{-1}\frac{d\alpha}{dt}\right)\rangle dt$$

because $\mathbf{w}(a) = 0, \mathbf{w}(b) = 0$.

First variation of arc length, cont.

Proposition

Let α be a regular curve in M. Then α is a critical point of the length functional if and only if

$$\left(\frac{d}{dt}\left(|\alpha'|^{-1}\frac{d\alpha}{dt}\right)\right)^T = 0.$$

α is a geodesic if and only if it is a critical point of the length functional and is parametrized proportional to arc length.

Proof: (Sketch) Since any vector field along α which vanishes at the end points is realized by a variation of α with end points fixed, we conclude that

$$\left(\frac{d}{dt}\left(|\alpha'|^{-1}\frac{d\alpha}{dt}\right)\right)^T = 0.$$

If $|\alpha'|$=constant, then we have $(\alpha'')^T = 0$. So it is a geodesic.

Another definition of geodesic

Recall that a geodesic is a regular curve so that (i) it is a critical point of the length functional; and (ii) it is parametrized proportional to arc length.
In case α only satisfies (i), we then have

$$(\alpha'')^T = -\left(|\alpha'|\frac{d}{dt}|\alpha'|^{-1}\right)\alpha'.$$

A regular curve α is said to be a pre-geodesic if $(\alpha'')^T$ is proportional to its tangent vector α'. That is:

$$(\alpha'')^T = \lambda\alpha'$$

for some smooth function $\lambda(t)$.

Equation for pre-geodesic in local coordinates

Suppose in local coordinates, $\alpha(t) = \mathbf{X}(u^1(t), u^2(t))$. Then

$$(\alpha'')^T = (\ddot{u}^k + \Gamma^k_{ij}\dot{u}^i\dot{u}^j)\mathbf{X}_k, \quad \alpha' = \dot{u}^k\mathbf{X}_k.$$

Hence the pre-geodesic equation is of the form:

$$\ddot{u}^k + \Gamma^k_{ij}\dot{u}^i\dot{u}^j = \lambda\dot{u}^k$$

for $k = 1, 2$.

Basic facts on calculus of variation

Consider the so-called action:

$$S = \int_a^b \mathcal{L}(t, \phi, \dot{\phi}) dt$$

Here $\phi = (\phi^1, \ldots, \phi^m)$ is a vector valued function of t, $\dot{\phi} = \frac{d}{dt}\phi$. Substitute ϕ for u, $\dot{\phi}$ for z,
$\mathcal{L} = \mathcal{L}(t; u^1, \ldots, u^m; z^1, \ldots, z^m)$ is called Lagrangian. We always assume that $\mathcal{L}$ is smooth in t, u, z in the domain under consideration.

$$\mathcal{L}(t, \phi, \dot{\phi}) = \mathcal{L}(t; \phi^1, \ldots, \phi^m; \dot{\phi}^1, \cdots, \dot{\phi}^m).$$

Example

Consider m particles in three space with coordinates (x^j, y^j, z^j) with mass $\mathfrak{m}_j$. Consider

$$\mathcal{L} = \frac{1}{2}\sum_j \mathfrak{m}_j \left[(\dot{x}^j)^2 + (\dot{y}^j)^2 + (\dot{z}^j)^2\right] - V(t, x, y, z)$$

where V is the potential energy. Here ϕ^k are those x^j, y^j, z^j; $\dot{\phi}^k$ are those $\dot{x}^j$, etc. They are functions of t along the trajectory of the particles.

Variation

Instead of writing $\frac{\partial}{\partial u^i}\mathcal{L}$, we write

$$\frac{\partial}{\partial \phi^i}\mathcal{L}$$

etc. Let us take a variation of the action. Namely, let $\eta(t) = (\eta^1(t), \dots, \eta^m(t))$ is a smooth (vector valued) function so that $\eta = 0$ near a, b. Let

$$S(\epsilon) = \int_a^b \mathcal{L}(t, \phi + \epsilon\eta, \overset{\cdot}{\overbrace{(\phi + \epsilon\eta)}})dt$$

Euler-Lagrangian equations

Suppose $\mathcal{L}(t, \phi + \epsilon\eta, \overbrace{(\phi + \epsilon\eta)}^{\cdot})$ is smooth for ϵ is small. Then

$$\frac{d}{d\epsilon}S(\epsilon)|_{\epsilon=0} = \int_a^b \left(\sum_k \eta^k \frac{\partial \mathcal{L}}{\partial \phi^k} + \sum_{k,\mu} \dot{\eta}^k \frac{\partial \mathcal{L}}{\partial \dot{\phi}^k} \right) dt$$
$$= \int_a^b \left(\sum_k \eta^k \left(\frac{\partial \mathcal{L}}{\partial \phi^k} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}^k} \right) \right) \right) dx.$$

Let

$$E_k =: \frac{\partial \mathcal{L}}{\partial \phi^k} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}^k} \right)$$

for $k = 1, \ldots, m$. These are called Euler-Lagrange expression (E.-L. expression).

Remark

As far as $S'(0)$ is concerned, instead of consider $\phi(t) + \epsilon\eta(t)$, it is equivalent to consider smooth variation $\phi(s,t)$, $|s| < \delta$, satisfying the following

- $\phi(0,t) = \phi(t)$, the original function;
- $\phi(s,a) = \phi(a), \phi(s,b) = \phi(b)$ for all s; i.e., the values are fixed at end points.
- Then $\phi(s,t) = \phi(t) + s\eta(t) + O(s^2)$ with $\eta(a) = \eta(b) = 0$, where $\eta(t) = \frac{\partial}{\partial s}\phi(s,t)|_{s=0}$.

Euler-Lagrangian equations, continued

Lemma

Let $f = (f_1, \dots, f_m)$ *be a vector valued continuous functions on* $[a, b]$ *such that*

$$\int_a^b \sum_k f_k \eta_k dt = 0$$

for any smooth functions η_k *with compact supports in* (a, b), *i.e.* $\eta_k = 0$ *near* a, b. *Then* $f_k = 0$ *for all* k.

Euler-Lagrangian equations, continued

ϕ is said to be an extremal of the action S mentioned above, if for any variation as above, we have $S'(0) = 0$.

Theorem

A C^2 function $\phi = (\phi^1, \ldots, \phi^m)$ is an extremal of S if and only if it satisfies the E-L equations for $\mathcal{L}$ above: $E_k = 0$, i.e.

$$\frac{\partial \mathcal{L}}{\partial \phi^k} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}^k}\right) = 0$$

for $k = 1, \ldots, m$.

Example

Example: As before, consider m particles in three space with coordinates (x^j, y^j, z^j) with mass $\mathfrak{m}_j$. Let

$$\mathcal{L} = \frac{1}{2}\sum_j \mathfrak{m}_j \left[(\dot{x}^j)^2 + (\dot{y}^j)^2 + (\dot{z}^j)^2\right] - V(t, x, y, z)$$

where V is the potential energy. Here ϕ^k are those x^j, y^j, z^j which depend only on t. $\dot{\phi}^k$ are those $\dot{x}^j$, etc.
E.-L. expressions are given by

$$E_{1j} = -\frac{\partial V}{\partial x^j} - \mathfrak{m}_j \frac{d^2 x^j}{dt^2}; E_{2j} = -\frac{\partial V}{\partial y^j} - \mathfrak{m}_j \frac{d^2 y^j}{dt^2}; E_{3j} = -\frac{\partial V}{\partial z^j} - \mathfrak{m}_j \frac{d^2 z^j}{dt^2}.$$

Application to geodesic: energy of a curve

Let M be a regular surface and α be a smooth curve defined on $[a, b]$. Then *energy* of α is defined to by

$$E(\alpha) = \frac{1}{2}\int_a^b \langle \alpha', \alpha' \rangle dt. \tag{1}$$

$\langle \alpha', \alpha' \rangle$ is called the energy density.
Remark: With the above notation, $(\ell(\alpha))^2 \leq (b - a)E(\alpha)$, and equality holds if and only if α is parametrized proportional to arc length.

Application to geodesic: energy of a curve, cont.

Theorem

Suppose α *is a regular curve defined on* $[a, b]$. α *is an extremal of* E *if and only if* α *is a geodesic.*

Proof: Let $\alpha(s, t)$ be a variation of α with end points fixed. Let $E(s)$ be the energy of $\alpha_s(t) = \alpha(s, t)$. Then at $s = 0$

$$E'(s) = \int_a^b \langle \frac{\partial^2 \alpha}{\partial s \partial t}, \frac{\partial \alpha}{\partial t} \rangle = -\int_a^b \langle \mathbf{w}, \alpha'' \rangle dt$$

where $\mathbf{w} = \frac{\partial \alpha}{\partial s}|_{s=0}$. Hence α is an extremal if and only if $(\alpha'')^T = 0$. That is, a geodesic.

E-L equations are equivalent to geodesic equations

To find the E-L equations for the energy functional in local parametrization: $\mathbf{X}(u^1, u^2)$ with first fundamental form g_{ij}. Then Lagrangian of the energy functional is:

$$\mathcal{L} = \frac{1}{2} g_{ij} \dot{u^i} \dot{u^j}.$$

Then (denote $\frac{\partial f}{\partial u^k}$ by $f_{,k}$ etc):

$$\begin{cases} \frac{\partial}{\partial u^k} \mathcal{L} = \frac{1}{2} g_{ij,k} \dot{u^i} \dot{u^j} \\ \frac{\partial}{\partial \dot{u^k}} \mathcal{L} = g_{ik} \dot{u^i} \\ \frac{d}{dt} \left(\frac{\partial}{\partial \dot{u^k}} \mathcal{L} \right) = g_{ik} \ddot{u^i} + g_{ik,l} \dot{u^l} \dot{u^i} \end{cases}$$

Hence E-L equations are:

$$\frac{1}{2} g_{ij,k} \dot{u^i} \dot{u^j} - \left(g_{ik} \ddot{u^i} + g_{ik,l} \dot{u^l} \dot{u^i} \right) = 0.$$

for $k = 1, 2$.

E-L equations are equivalent to geodesic equations, cont.

$$g_{ik}\ddot{u}^i + g_{pk,q}\dot{u}^q\dot{u}^p - \frac{1}{2}g_{pq,k}\dot{u}^p\dot{u}^q = 0.$$

Hence

$$\ddot{u}^i + \frac{1}{2}g^{ik}\left(2g_{pk,q}\dot{u}^q\dot{u}^p - g_{pq,k}\dot{u}^p\dot{u}^q\right) = 0.$$

Or

$$\ddot{u}^i + \frac{1}{2}g^{ik}\left(g_{pk,q}\dot{u}^q\dot{u}^p + g_{qk,p}\dot{u}^p\dot{u}^q - g_{pq,k}\dot{u}^p\dot{u}^q\right) = 0.$$

Finally, we have

$$\ddot{u}^i + \Gamma^i_{pq}\dot{u}^p\dot{u}^q = 0.$$

Example

Consider the surface of revolution $u^1 \leftrightarrow u$, $u^2 \leftrightarrow v$:

$$\mathbf{X}(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$$

$f > 0$. We want to find the equations of geodesics.
Method 1: $g_{11} = f^2$, $g_{12} = 0$, $g_{22} = (f')^2 + (g')^2$. The Christoffel symbols are given by

$$\Gamma^1_{11} = 0, \Gamma^2_{11} = -\frac{ff'}{(f')^2 + (g')^2}, \Gamma^1_{12} = \frac{ff'}{f^2};$$

$$\Gamma^2_{12} = 0, \Gamma^1_{22} = 0, \Gamma^2_{22} = \frac{f'f'' + g'g''}{(f')^2 + (g')^2}.$$

Hence geodesic equations are

$$\ddot{u} + \frac{2ff'}{f^2}\dot{u}\dot{v} = 0$$

and

$$\ddot{v} - \frac{ff'}{(f')^2 + (g')^2}\dot{u}^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}\dot{v}^2 = 0.$$

Example, cont.

Method 2: On the other hand, the $\frac{1}{2}$ of the energy density of a curve is given by

$$\mathcal{L} = \frac{1}{2}(f^2(\dot{u})^2 + ((f')^2 + ((g')^2)(\dot{v})^2).$$

Then

$$\frac{\partial}{\partial u}\mathcal{L} = 0, \ \frac{\partial}{\partial v}\mathcal{L} = ff'\dot{u}^2 + (f'f'' + g'g'')\dot{v}^2;$$
$$\frac{\partial}{\partial \dot{u}}\mathcal{L} = f^2\dot{u}, \ \frac{\partial}{\partial \dot{v}}\mathcal{L} = ((f')^2 + ((g')^2)\dot{v}.$$

The E-L equations are:

$$\frac{d}{dt}(f^2\dot{u}) = 0,$$

and

$$ff'\dot{u}^2 + (f'f'' + g'g'')\dot{v}^2 - \frac{d}{dt}\left(((f')^2 + ((g')^2)\dot{v}\right) = 0$$

$$\frac{d}{dt}(f^2\dot{u}) = 0,$$

and

$$ff'\dot{u}^2 + (f'f'' + g'g'')\dot{v}^2 - \frac{d}{dt}\left(((f')^2 + ((g')^2)\dot{v}\right) = 0$$

Compare with previous computations:

$$\ddot{u} + \frac{2ff'}{f^2}\dot{u}\dot{v} = 0$$

and

$$\ddot{v} - \frac{ff'}{(f')^2 + (g')^2}\dot{u}^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}\dot{v}^2 = 0.$$

Example

We may also use the energy functional to compute Γ_{ij}^k. Consider the polar coordinates of the plane $\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, 0)$. Let $r \leftrightarrow u^1, \theta \leftrightarrow u^2$. Then $g_{11} = 1, g_{12} = 0, g_{22} = r^2$. Then $\frac{1}{2}$ of the energy density is given by

$$\mathcal{L} = \frac{1}{2}((\dot{r})^2 + r^2(\dot{\theta})^2).$$

Then

$$\frac{\partial}{\partial r}\mathcal{L} = r(\dot{\theta})^2, \ \frac{\partial}{\partial \dot{r}}\mathcal{L} = \dot{r};$$
$$\frac{\partial}{\partial \theta}\mathcal{L} = 0, \ \frac{\partial}{\partial \dot{\theta}}\mathcal{L} = r^2\dot{\theta}.$$

So E-L equations are

$$\begin{cases} r(\dot{\theta})^2 - \frac{d}{dt}\dot{r} = 0 \\ -\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0. \end{cases}$$

$$\begin{cases} \ddot{r} - r(\dot{\theta})^2 = 0 \\ \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = 0. \end{cases}$$

These are geodesic equations. Hence one can obtain Γ_{ij}^k by comparing with the geodesic equations.