1/12/22 MATH4030 Tutorial

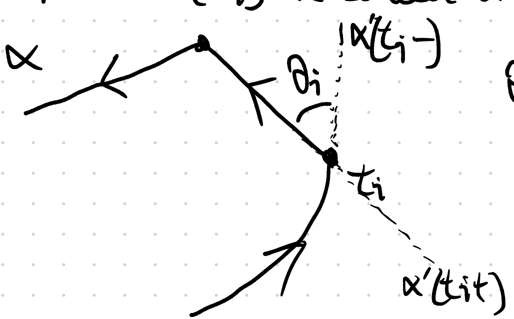
Reminders:

- Assignment 6 due tonight 11:59pm.
- Final: 9 Dec 1530-1730 @ University Gymnasium.

Recall Def:

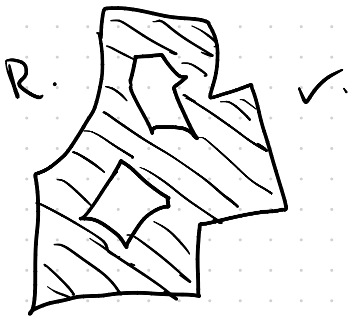
- $\alpha : [a,b] \to M$ is a piecewise regular, simple, closed curve if:
  - simple: $\forall t_1, t_2 \in (a,b)$, $t_1 \neq t_2$, $\alpha(t_1) \neq \alpha(t_2)$.
  - closed: $\alpha(a) = \alpha(b)$.
  - piecewise regular: $\exists\, a = t_0 < t_1 < \dots < t_k < t_{k+1} = b$ s.t. α is differentiable and regular on each (t_i, t_{i+1}) $i = 0, \dots, k$.

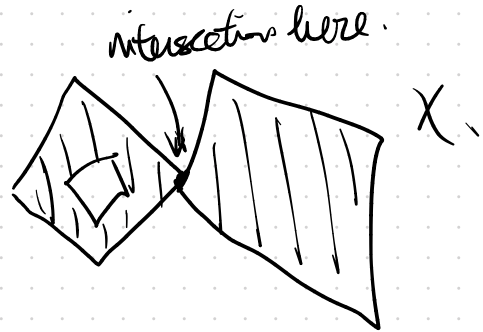
Each $\alpha(t_i)$ is called a vertex of α. At each vertex, we have an exterior angle θ_i.



$\theta_i \in (-\pi, \pi]$.

- M is a regular surface. A connected region $R \subset M$ is regular, if R is compact and has as boundary ∂R = the finite union of simple, closed, piecewise regular curves which do not intersect each other.

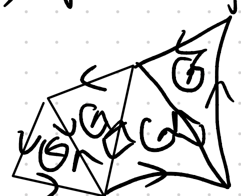




- A triangulation $\mathcal{T}$ of R is a finite family of triangles T_i s.t.

1) $R = \bigcup_{i=1}^{n} T_i$

2) if $T_i \cap T_j \neq \emptyset$, then they either have a common edge or a common vertex only.



$$\chi(\mathcal{T}) = \underbrace{F}_{\#\text{ of triangles}} - \overbrace{E}^{\#\text{edges}} + \underbrace{V}_{\#\text{vertices}}$$

– Euler-Poincaré Characteristic of $\mathcal{T}$.

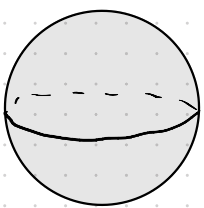
Facts:

1) Every regular region R admits a triangulation
2) χ for R doesn't depend on choice of triangulation, so $\chi(R)$ is well-defined.
3) Classification of compact surfaces: $M \subset \mathbb{R}^3$ compact connected surface, then M is homeomorphic to a sphere w/ a number of handles g (genus) attached. and.

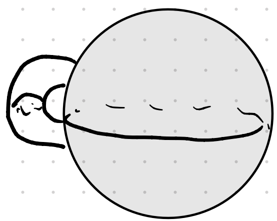
$$\chi(M) = 2-2g. \qquad g = 0, 1, 2, \dots$$

Topology!

4) If $\chi(M_1) = \chi(M_2)$, then $M_1 \simeq M_2$.

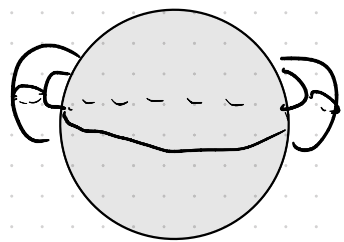


$\chi(S^2) = 2$,



$\simeq$

$\chi(T) = 0$,



$\simeq$

$\chi = -2$, ...

Global Gauss-Bonnet Thm: Let $R \subset M$ be a regular region of an oriented surface, and let $C_1, C_2, \dots, C_n$ be the simple, closed, piecewise regular curves forming the boundary ∂R, with each positively oriented, and let $\theta_1, \dots, \theta_p$ be the exterior angles, then

$$\sum_{i=1}^{n} \int_{C_i} k_g(s)\,ds + \iint_R K\,d\sigma + \sum_{i=1}^{p} \theta_p = 2\pi \chi(R).$$

$\sqrt{EG-F^2}\,du\,dv$

(when M is compact, oriented, no boundary).

Cor: We can view M as a regular region w/ no boundary (ie. $\partial M = \phi$), so we get:

$$\iint_M K\,d\sigma = 2\pi \chi(M).$$

App1: Any compact surface of everywhere positive curvature is homeomorphic to the sphere.

Pf: Since $K>0$, $0 < \iint_R K\,d\sigma = 2\pi\chi(M) \Rightarrow \chi(M) > 0$.

S^2 is the only compact surface w/ positive χ. $\Rightarrow \chi(M) = \chi(S^2)$

$\Rightarrow M \simeq S^2$. //

App2: Let M be compact, oriented, regular surface in $\mathbb{R}^3$ not homeomorphic to S^2.
Then show that K achieves both positive and negative values.

Pf1: Since $M \not\simeq S^2$, $\chi(M) \leq 0 \Rightarrow \int_M K\,d\sigma \leq 0$., so that means K cannot be everywhere positive.

Since M is compact, by earlier in semester, we know that M contains at least one elliptic point p_0, where $K(p_0) > 0$. So K cannot be negative everywhere either

Pf 2: Look at $f(p) = \|p\|^2$ ...

do Carmo 4-6/4-7.

One further application: an orientable compact surface M has a differentiable vector field w/ no singular points iff M is homeomorphic to the torus.