Existence and uniqueness theorems in ODE

Ref: *Ordinary differential equations, Birkoff and Rota*

Let $A(t) = (a_{ij}(t))_{n\times n}$ be a smooth family $n \times n$ matrix, $t \in [a, b]$. Consider the following initial valued problem (IVP): Given A and a constant $\mathbf{x}_0 \in \mathbb{R}^n$, to find $\mathbf{x} : [a, b] \to \mathbb{R}^n$ satisfying:

$$\begin{cases} \mathbf{x}'(t) & = A(t)\mathbf{x}(t), \quad t \in [a, b]; \\ \mathbf{x}(a) & = \mathbf{x}_0. \end{cases}$$

Theorem

Given any $\mathbf{x}_0 \in \mathbb{R}^n$*, the exists a unique solution of the above IVP.*

[Proof.](Sketch) For simplicity let us assume $a = 0$.
Existence: Define inductively, with $\mathbf{x}_0(t) = \mathbf{x}_0$ for all t, and

$$\mathbf{x}_{k+1}(t) = \mathbf{x}_0 + \int_0^t A(\tau)\mathbf{x}_k(\tau)d\tau.$$

for $k \geq 0$.
Let $M = \sup_{t\in[a,b]} ||A||(t)$ and $||A(t)||^2 = \mathrm{tr}(AA^T(t))$. For $k \geq 1$, we have

$$|\mathbf{x}_{k+1}(t) - \mathbf{x}_k(t)| \leq M \int_0^t |\mathbf{x}_k(\tau) - \mathbf{x}_{k-1}(\tau)|d\tau.$$

Inductively, we have (why?)

$$\begin{aligned}
&|\mathbf{x}_{k+1}(t) - \mathbf{x}_k(t)| \\
\le & M^k \int_0^t \int_0^{\tau_{k-1}} \cdots \int_0^{\tau_2} \int_0^{\tau_1} |\mathbf{x}_1(\tau_1) - \mathbf{x}_0(\tau_1)| d\tau_1 d\tau_2 \ldots d\tau_{k-1} d\tau_k \\
\le & \frac{M^k b^k S}{k!}
\end{aligned}$$

where integration is over the domain $t \ge \tau_k \ge \cdots \ge \tau_1$ and $S = \sup_{t\in[0,b]} |\mathbf{x}_1(t) - \mathbf{x}_0(t)|$.
Hence $\sum_{k=1}^{\infty} |\mathbf{x}_{k+1}(t) - \mathbf{x}_k(t)| \le C$ for some constant C for all $t \in [0, b]$. This implies that $\mathbf{x}_k \to \mathbf{x}_\infty$ uniformly on $[0, b]$ which satisfies:

$$\mathbf{x}_\infty(t) = \mathbf{x}_0 + \int_0^t A(\tau)\mathbf{x}_\infty(\tau) d\tau,$$

(why?) Now $\mathbf{x}_\infty$ is the solution of the above IVP.

Proof.

Uniquess: Sufficient to prove that if $\mathbf{x}_0 = \mathbf{0}$, then any solution must be trivial. So let $\mathbf{x}$ be such a solution, then

$$\frac{d}{dt}||\mathbf{x}||^2 = 2\langle A\mathbf{x}, \mathbf{x}\rangle \leq 2M||\mathbf{x}||^2.$$

Hence

$$\frac{d}{dt}\left(\exp(-2Mt)||\mathbf{x}||^2\right) \leq 0.$$

This will imply that $||\mathbf{x}||^2 \equiv 0$. (Why?) □

Fundamental theorem for curves in $\mathbb{R}^3$

Theorem

Let $\kappa(s) > 0$ and $\tau(s)$ be smooth function on (a, b). There exists a regular curve $\alpha : (a, b) \to \mathbb{R}^3$ with $|\alpha'| = 1$, such that the curvature and torsion of α are k, τ respectively.
Moreover, α is unique in the sense: If β is another curve satisfying the above conditions, then $\beta(s) = \alpha(s)P + \vec{c}$ for some constant orthogonal matrix P and some constant vector $\vec{c}$. **Here α, β are considered as row vectors.**

Existence

Let

$$A(s) = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}.$$

Let $X(s)$ be the 3×3 matrix and fix s_0 which is the solution of:

$$\begin{cases} X' & = AX \quad \text{in } (a, b); \\ X(s_0) & = I. \end{cases}$$

The solution exists by a theorem in ODE.

X is orthogonal.

$$(X^tX)' = (X^t)'X + X^tX' = (AX)^tX + X^tAX = X^tA^tX + X^tAX = 0$$

because $A^t = -A$. Hence $X^tX = I$ because $X^t(s_0)X(s_0) = I$. (**Using $(XX^t)'$ may be more involved.**) Hence $X(s)$ is orthogonal. Since $\det X(s) = 1$ or -1 and initially, $\det X(s_0) = 1$, we have $\det X(s) = 1$.

Write

$$X = \begin{pmatrix} \widetilde{T} \\ \widetilde{N} \\ \widetilde{B} \end{pmatrix}.$$

Define $\alpha(s) = \int_{s_0}^{s} \widetilde{T}(\sigma)d\sigma$. Let T, N, B be the tangent, principal normal and binormal of α, and let $\kappa_\alpha, \tau_\alpha$ be the curvature and torsion of α.

- $\alpha' = \widetilde{T}$ which has length 1. So $T = \widetilde{T}$.
-
$$\kappa_\alpha N = T' = \widetilde{T}' = \kappa \widetilde{N}.$$
  we have $\kappa_\alpha = \kappa$ and $N = \widetilde{N}$.
- Since $\widetilde{T}, \widetilde{N}, \widetilde{B}$ are positively oriented, we conclude that
$$B = T \times N = \widetilde{T} \times \widetilde{N} = \widetilde{B},$$
  and
$$-\tau_\alpha N = B' = \widetilde{B}' = -\tau \widetilde{N} = -\tau N.$$

Uniqueness

Lemma

Let α be a regular curve parametrized by arc length with Frenet frame $\{T, N, B\}$ and with curvature and torsion κ, τ. Let P be an orthogonal matrix with determinant 1 and let $\beta = \alpha P + \vec{c}$, where $\vec{c}$ is a constant vector. Then the Frenet frame of β is TP, NP, BP with same curvature and torsion.

Proof: Exercise.

Proof of Uniqueness

Uniqness: Let α, β as in the theorem. Let $T_\alpha, N_\alpha, B_\alpha$ be the unit tangent, principal normal, binormal of α; and let $T_\beta, N_\beta, B_\beta$ be the unit tangent, principal normal, binormal of β. Fix $s_0 \in (a, b)$. Let P be an orthogonal matrix with determinant 1 such that

$$\begin{pmatrix} T_\beta(s_0) \\ N_\beta(s_0) \\ B_\beta(s_0) \end{pmatrix} = \begin{pmatrix} T_\alpha(s_0) \\ N_\alpha(s_0) \\ B_\alpha(s_0) \end{pmatrix} P.$$

Here $T_\alpha, \dots,$ etc are considered as row vectors. Let $\gamma(s) = \alpha(s)P$. Let T_γ, N_γ, B_γ be unit tangent, principal normal, binormal of γ.

Then

$$T_\gamma = \gamma' = \alpha' P = T_\alpha P,$$

$$\kappa N_\gamma = T_\gamma' = T_\alpha' P = \kappa N P.$$

and so $T_\gamma = T_\alpha P, N_\gamma = N_\alpha P$. Hence $B_\gamma = B_\alpha P$. We have

$$\begin{pmatrix} T_\gamma \\ N_\gamma \\ B_\gamma \end{pmatrix}' = \begin{pmatrix} T_\alpha \\ N_\alpha \\ B_\alpha \end{pmatrix}' P = A \begin{pmatrix} T_\alpha \\ N_\alpha \\ B_\alpha \end{pmatrix} P = A \begin{pmatrix} T_\gamma \\ N_\gamma \\ B_\gamma \end{pmatrix}$$

where A is as above. Since

$$\begin{pmatrix} T_\gamma(s_0) \\ N_\gamma(s_0) \\ B_\gamma(s_0) \end{pmatrix} = \begin{pmatrix} T_\alpha(s_0) \\ N_\alpha(s_0) \\ B_\alpha(s_0) \end{pmatrix} P = \begin{pmatrix} T_\beta(s_0) \\ N_\beta(s_0) \\ B_\beta(s_0) \end{pmatrix}.$$

we have $T_\gamma = T_\beta$, by uniqueness theorem of ODE. So $\gamma(s) + \vec{c} = \beta(s)$ for some constant vector $\vec{c}$. That is: $\beta(s) = \alpha(s) P + \vec{c}$.

Geometric meaning of curvature

Proposition

Let $\alpha(s)$ be a plane curve parametrized by arc length defined on (a, b). Let $s_0 \in (a, b)$. Suppose $\kappa(s_0) > 0$. Then the following are true:

(i) *For any $s_1 < s_2 < s_3$ sufficiently close to s_0, $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ are not collinear.*

(ii) *For $s_1 < s_2 < s_3$ sufficiently close to s_0 so that $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ are not collinear,*

(iii) *Let $c(s_1, s_2, s_3)$ be the center of the unique circle $C(s_1, s_2, s_3)$ passing through $\alpha(s_1), \alpha(s_2), \alpha(s_3)$.*

As $s_1, s_2, s_3 \to s_0$, $C(s_1, s_2, s_3)$ will converge to a circle passing through $\alpha(s_0)$ tangent to α at $\alpha(s_0)$ with radius $1/\kappa(s_0)$.

[Proof] (i) Suppose $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ lie on a straight line. Then

$$\langle \alpha(s_i) - \vec{v}, \vec{n} \rangle = 0$$

for some constant vectors $\vec{v}, \vec{n}$ with $|\vec{n}| = 1$, for $i = 1, 2, 3$. Let $f(s) = \langle \alpha(s) - \vec{v}, \vec{n} \rangle$. Then $f(s_i) = 0$ for $i = 1, 2, 3$. Hence $f'(\xi_1) = f'(\xi_2) = 0$ for some $s_1 < \xi_1 < s_2 < \xi_2 < s_3$ and $f''(\eta) = 0$ for some $\xi_1 < \eta < \xi_2$. That is:

$$\begin{cases} \langle \alpha'(\xi_1), \vec{n} \rangle = & \langle \alpha'(\xi_2), \vec{n} \rangle = 0; \\ \langle \alpha''(\eta), \vec{n} \rangle = & 0. \end{cases}$$

As $s_1, s_2, s_3 \to s_0$, $\vec{n} \to N(s_0)$ and $\alpha''(\eta) = \kappa(s_0) N(s_0)$. This implies $\kappa(s_0) = 0$. Contradiction.

(ii) Let $C(s_1, s_2, s_3)$ be given by

$$||\mathbf{x} - c|| = r.$$

where $c = c(s_1, s_2, s_3)$.
Let $h(s) = ||\alpha(s) - c||^2$. Then $h(s_i) = r^2$ for $i = 1, 2, 3$. Hence $h'(\xi_1) = h'(\xi_2) = 0$ for some $s_1 < \xi_1 < s_2 < \xi_2 < s_3$ and $h''(\eta) = 0$ for some $\xi_1 < \eta < \xi_2$. Hence

$$\begin{cases} \langle \alpha'(\xi_1), \alpha(\xi_1) - c \rangle & = \langle \alpha'(\xi_2), \alpha(\xi_2) - c \rangle = 0; \\ \langle \alpha''(\eta), \alpha(\eta) - c \rangle + 1 & = 0. \end{cases}$$

If $c \to c_\infty$ for some sequence $s_1 < s_2 < s_3 \to s_0$, then

$$\langle \alpha'(s_0), \alpha(s_0) - c_\infty \rangle = 0, \quad \langle \alpha''(s_0), \alpha(s_0) - c_\infty \rangle = -1$$

So $c_\infty - \alpha(s_0) = \frac{1}{\kappa(s_0)} N(s_0)$. From this the result follows.
The limiting circle is called the *osculating circle*.