Minimal surfaces: definition

Definition

A regular surface M is said to be minimal if the mean curvature of M is identically zero.

Proposition

For a graph $\mathbf{X}(x, y) = (x, y, f(x, y))$.
Minimal if

$$0 = H = \frac{1}{2} \cdot \frac{(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy}}{(1 + f_x^2 + f_y^2)^{\frac{3}{2}}}.$$

Or

$$\mathrm{div}\left(\frac{\nabla f}{1 + |\nabla f|^2}\right) = 0.$$

Minimal surfaces in isothermal coordinates

Definition

Let $\mathbf{X}(u,v)$ be a local parametrization of a regular surface. $\mathbf{X}$ is said to be isothermal if $|\mathbf{X}_u| = |\mathbf{X}_v| = \lambda$, and $\langle \mathbf{X}_u, \mathbf{X}_v \rangle = 0$.

To check whether a surface is minimal, the following fact is useful.

Proposition

Let $\mathbf{X}(u,v)$ be an isothermal coordinate parametrization of a regular surface M. Let $\mathbf{N} = \mathbf{X}_u \times \mathbf{X}_v / |\mathbf{X}_u \times \mathbf{X}_v|$. Then

$$\mathbf{X}_{uu} + \mathbf{X}_{vv} = 2\lambda^2 H\mathbf{N}$$

where H is the mean curvature.

Proof

Proof.

$$\langle \mathbf{X}_{uu}, \mathbf{X}_u \rangle = \frac{1}{2} \langle \mathbf{X}_u, \mathbf{X}_u \rangle_u = \lambda_u.$$

$$\langle \mathbf{X}_{vv}, \mathbf{X}_u \rangle = -\langle \mathbf{X}_v, \mathbf{X}_{uv} \rangle = -\lambda_u.$$

So

$$\langle \mathbf{X}_{uu} + \mathbf{X}_{vv}, \mathbf{X}_u \rangle = 0.$$

Similarly, $\langle \mathbf{X}_{uu} + \mathbf{X}_{vv}, \mathbf{X}_v \rangle = 0$. Hence

$$\mathbf{X}_{uu} + \mathbf{X}_{vv} = \langle \mathbf{X}_{uu} + \mathbf{X}_{vv}, \mathbf{N} \rangle \mathbf{N} = (e+g)\mathbf{N} = 2\lambda^2 H \mathbf{N},$$

because

$$H = \frac{1}{2} \frac{eG - 2fF + Eg}{EG - F^2} = \frac{1}{2} \frac{e+g}{\lambda^2}.$$

Corollary

Suppose $\mathbf{X}(u, v)$ is an an isothermal coordinate parametrization of a regular surface M. M is a minimal surface if and only if $\mathbf{X}_{uu} + \mathbf{X}_{vv} = 0$. (That is: each coordinate function is harmonic as a function of u, v.)

Minimal surfaces and complex variables

This is only for your reference: Let $\mathbf{X}(u,v)$ be a coordinate parametrization of M. Let $\phi_1 = x_u - \sqrt{-1}x_v$, $\phi_2 = y_u - \sqrt{-1}y_v$, $\phi_3 = z_u - \sqrt{-1}z_v$. Then

(i) $\mathbf{X}$ is isothermal if and only if $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$.

(ii) M is minimal if and only if ϕ_i are analytic for $i = 1, 2, 3$.

Examples

- A plane is a minimal surface.
- Let M be the catenoid: the surface of revolution by rotating the curve $(a\cosh v, 0, v)$ about the z-axis. Take $a = 1$

$$\mathbf{X}(u, v) = (\cosh v \cos u, \cosh v \sin u, v).$$

Then $E = G = \cosh^2 v$, $F = 0$.

$$\mathbf{X}_{uu} = (-(\cosh v \cos u, -\cosh v \sin u, 0);$$

$$\mathbf{X}_{vv} = (\cosh v \cos u, \cosh v \sin u, 0).$$

So $\mathbf{X}_{uu} + \mathbf{X}_{vv} = 0$. Catenoid is minimal.

Surfaces of revolution which are minimal

Consider the surface of revolution given by

$$\mathbf{X}(u,v) = (f(v)\cos u, f(v)\sin u, g(v)); (f')^2 + (g')^2 = 1$$

It is minimal if and only if

$$0 = H = \frac{1}{2}\frac{-g' + f(g'f'' - g''f')}{f}.$$

Suppose $g' \neq 0$ somewhere, then v can be expressed as a function of z and $f(v) = \phi(g(v))$. We have $\dot\phi$ means derivative w.r.t. z etc.

$$f' = \dot\phi g', \quad f'' = \ddot\phi (g')^2 + \dot\phi g''.$$

So we have

$$0 = -g' + \phi\left(g'(\ddot\phi(g')^2 + \dot\phi g'') - g''\dot\phi g'\right) = -g' + \phi\ddot\phi(g')^3$$

Surfaces of revolution which are minimal, cont.

So

$$-1 + \phi\ddot{\phi}(g')^2 = 0.$$

Since $(f')^2 + (g')^2 = 1$, so $(g')^2(1 + \dot{\phi}^2) = 1$, and we have

$$\frac{\phi\ddot{\phi}}{1 + \dot{\phi}^2} = 1.$$

Check, $\phi = a\cosh((z + c)/a)$ are solutions.
Hence $g' \neq 0$ and the surface is part of a catenoid, or $g' \equiv= 0$, then the surface is a part of a plane.

First variational formula for area: Minimal surfaces are critical points of the areas functional

Let $\mathbf{X} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a coordinate parametrization of a regular surface M. Let $\overline{D}$ be a compact domain in U and let $Q = \mathbf{X}(D) \subset M$. Let $h(u, v)$ be a smooth function on $\overline{D}$. Let $\mathbf{N} = \mathbf{X}_u \times \mathbf{X}_v / |\mathbf{X}_u \times \mathbf{X}_v|$ be the unit normal of the surface. Define:

$$\mathbf{Y}(u, v; t) = \mathbf{X}(u, v) + th(u, v)\mathbf{N}(u, v).$$

Lemma

There exists $\epsilon > 0$ such that for each fixed t with $|t| < \epsilon$, $\mathbf{Y}(u, v; t)$ represent a parametrized regular surface. ($\mathbf{Y}(u, v; t)$ is called a **normal variation** *of $\overline{Q}$.)*

Proof

Let $\mathbf{Y}_u = \mathbf{X}_u + t(h_u\mathbf{N} + h\mathbf{N}_u)$, etc. So

$$\begin{aligned}\mathbf{Y}_u \times \mathbf{Y}_v =& \mathbf{X}_u \times \mathbf{X}_v + t\left[(h_u\mathbf{N} + h\mathbf{N}_u) \times \mathbf{X}_v + \mathbf{X}_u \times (h_v\mathbf{N} + h\mathbf{N}_v)\right] \\ &+ t^2(h_u\mathbf{N} + h\mathbf{N}_u) \times (h_u\mathbf{N} + h\mathbf{N}_u) \\ =& \mathbf{X}_u \times \mathbf{X}_v + R(u, v, t).\end{aligned}$$

Since $|\mathbf{X}_u \times \mathbf{X}_v| \geq C_1$ for some $C_1 > 0$ on $\overline{D}$ and $|R| \leq \epsilon C_2$ for some $C_2 > 0$ on $\overline{D}$ independent of ϵ. So $\mathbf{Y}_u \times \mathbf{Y}_v \neq \mathbf{0}$ if ϵ is small enough.

First variational formula, cont.

Let $\epsilon > 0$ be as above. Define $A(t)$ to be the area of

$$M(t) = \{\mathbf{Y}(u,v,t) | (u,v) \in \overline{D}\}.$$

Theorem (First variation of area)

$$\frac{dA}{dt}\Big|_{t=0} = -2 \iint_{\overline{Q}} hH dA$$

where H is the mean curvature of M. Here for any function ϕ on $\overline{D}$,

$$\iint_{\overline{Q}} \phi dA := \iint_{\overline{D}} \phi |\mathbf{X}_u \times \mathbf{X}_v| dudv.$$

Proof

Proof: Let $E(u,v,t) = \langle \mathbf{Y}_u(u,v,t), \mathbf{Y}_u(u,v,t) \rangle$ etc. Let $E_0(u,v) = E(u,v,0)$ etc (which are the coefficients of the first fundamental form of $\mathbf{X}$).

$$\begin{aligned} E(u,v,t) =& E_0(u,v) + 2th(u,v)\langle \mathbf{N}_u, \mathbf{X}_u \rangle + O(t^2) \\ =& E_0(u,v) - 2th(u,v)e(u,v) + O(t^2); \\ F(u,v,t) =& F_0(u,v) + 2th(u,v)\langle \mathbf{N}_u, \mathbf{X}_v \rangle + O(t^2) \\ =& F_0(u,v) - 2th(u,v)f(u,v) + O(t^2); \\ G(u,v,t) =& G_0(u,v) + 2th(u,v)\langle \mathbf{N}_v, \mathbf{X}_v \rangle + O(t^2) \\ =& G_0(u,v) - 2th(u,v)g(u,v) + O(t^2), \end{aligned}$$

where e, f, g are the coefficients of the second fundamental form of $\mathbf{X}$. Hence

$$EG - F^2 = E_0 G_0 - F_0^2 - 2t\left(eG_0 - 2fF_0 + gG_0\right) + O(t^2).$$

First variational formula, cont.

Hence

$$
\begin{aligned}
A(t) &= \iint_{\overline{D}} \sqrt{(EG - F^2)}dudv \\
&= \iint_{\overline{D}} \sqrt{E_0 G_0 - F_0^2} dudv - t \iint_{\overline{D}} h \frac{eG_0 - 2fF_0 + gG_0}{\sqrt{E_0 G_0 - F_0^2}} dudv \\
&\quad + O(t^2) \\
&= \iint_{\overline{D}} \sqrt{E_0 G_0 - F_0^2} dudv - 2t \iint_{\overline{Q}} hHdA + O(t^2).
\end{aligned}
$$

Corollary

$A'(0) = 0$ for all normal variation of $\overline{Q}$ if and only if $H \equiv 0$ on Q. Actually, a regular surface M is minimal if and only if $A'(0) = 0$ for all normal variation of M with compact support: i.e. any variation by $f\mathbf{N}$ where f has satisfies $\overline{f \neq 0}$ is a compact set in M.

Construction of bump function

To prove the theorem, we need to construct a so-called *bump function*, starting with

$$\phi(t) = \begin{cases} 0, & \text{if } t \le 0; \\ e^{-\frac{1}{t}}, & \text{if } t > 0. \end{cases}$$

Consider the function:

$$\Phi(t) = \frac{\psi_1(t)}{\psi_1(t) + \psi_2(t)}$$

where

$$\psi_1(t) = \phi(2+t)\phi(2-t), \psi_2(t) = \phi(t-1) + \phi(-1-t).$$

Then $\Phi(t)$ satisfies $\Phi(t) \ge 0$, and

$$\Phi(t) = \begin{cases} 1, & \text{if } |t| \le 1; \\ 0, & \text{if } |t| \ge 2. \end{cases}$$

A general result

Lemma

Let h be a smooth function defined in a domain $U \subset \mathbb{R}^2$. Suppose

$$\iint_U f\, h du dv = 0$$

for all smooth function f with compact support in U, then $h \equiv 0$.

A reference for minimal surfaces: *Osserman, A survey of minimal surfaces.*

Constant mean curvature surfaces

Let M be an regular surface which is the boundary of a domain. Let $\mathbf{N}$ be a unit normal vector field. Consider the variation given by variational vector field $f\mathbf{N}$: Namely in local coordinate patch:

$$\mathbf{Y}(u,v;t) = \mathbf{X}(u,v) + tf\mathbf{N}(u,v).$$

Or in general $\mathbf{Y} = \mathbf{X} + tf\mathbf{N}$ where $\mathbf{X}$ is the position vector of a point in M.

Variation with constraint

We want to compute the variation of the area under the constraint that the volume is fixed.

As before, let $A(t)$ be the area of the surface $\mathbf{Y}(t)$. Then we have

$$A'(0) = -2 \iint_M fH dA.$$

Volume constraint

Let $V(t)$ be the volume contained insider $\mathbf{Y}(t)$. So f must be such that $V'(0) = 0$.
Let $\mathbf{X}(u, v)$ be a local parametrization from $U \to M \subset \mathbb{R}^3$.
Consider the map

$$\mathbf{F}(u, v, w) = \mathbf{X}(u, v) + w\mathbf{N}(u, v) = (x, y, z).$$

Then the volume between $\mathbf{X}(u, v)$ and $\mathbf{Y}(u, v, t)$ is given by

$$V(t) = \iint_U \left(\int_0^{tf(u,v)} J \, dw \right) dudv$$

where

$$\begin{aligned} J =& \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} \\ =& \mathbf{F}_u \times \mathbf{F}_v \cdot \mathbf{F}_w \\ =& (\mathbf{X}_u + w\mathbf{N}_u) \times (\mathbf{X}_v + w\mathbf{N}_v) \cdot \mathbf{N} \\ =& \mathbf{X}_u \times \mathbf{X}_v \cdot \mathbf{N} + O(w) \\ =& |\mathbf{X}_u \times \mathbf{X}_v| + O(w). \end{aligned}$$

Hence

$$V(t) = t \iint_U f|\mathbf{X}_u \times \mathbf{X}_v| dudv + O(t^2) and$$

$$V'(0) = \iint_U f|\mathbf{X}_u \times \mathbf{X}_v| dudv.$$

Theorem

Let M be as above. Suppose M is a critical point of the area functional under normal variation which preserves volume. Then M has constant curvature.

Proof.

From above, we have

$$\iint_M fHdA = 0$$

for all f satisfying $\iint_M fdA = 0$. Hence H must be constant. In fact, let a be the average of H over M: $a = \frac{1}{A(M)} \iint_M HdA$. Then

$$\iint_M f(H-a)dA = 0$$

for all f satisfying $\iint_M fdA = 0$. Let $f = H - a$, then $\iint_M fdA = 0$. Hence

$$\iint_M (H-a)^2 dA = 0.$$

Hence $H \equiv a$ which is a constant. □

Delaunay surfaces

For your reference.

Theorem

(Delaunay). A complete immersed surface of revolution of constant mean curvature is a roulette of a conic.

- Roulette of a circle gives a circular cylinder.
- Roulette of a parabola gives a catenoid.
- Roulette of an ellipse is called an undulary and it gives an unduloid.
- Roulette of a hyperbola is called a nodary and it gives a nodoid.

Ref: Opera's book, section 3.6