THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2022-23 Term 1

Solution to Homework 9

1. If P and Q are projections of a Hilbert space H onto closed subspaces E and E', respectively, and $PQ = QP$, then
$$P + Q - PQ$$
is a projection of H onto $E + E'$.

Proof. By $PQ = QP$,
$$Q(I - P) = Q - QP = Q - PQ = (I - P)Q.$$
Write $A := P + Q - PQ$. Note that $P^2 = P$, and so $P(I - P) = (I - P)P = 0$. Then
$$\begin{aligned} A^2 &= (P + (Q - PQ))(P + (Q - PQ)) \\ &= P^2 + P(Q - PQ) + (Q - PQ)P + (Q - PQ)(Q - PQ) \\ &= P + P(I - P)Q + Q(I - P)P + Q^2(I - P)^2 \\ &= P + 0 + 0 + Q(I - P) \\ &= P + Q - PQ = A. \end{aligned}$$
Hence A is a projection of H.

Since $AH = PH + Q(I - P)H \subset PH + QH$, we have $\operatorname{Im} A \subset \operatorname{Im} P + \operatorname{Im} Q$. On the other hand, let $x \in E + E'$. Then $x = x_1 + x_2$ for some $x_1 \in E$ and $x_2 \in E'$. By $H = \operatorname{Im} P \oplus \operatorname{Ker} P = \operatorname{Im} P \oplus \operatorname{Im}(I - P)$, we have $x_2 = y_1 + y_2$ for some $y_1 \in \operatorname{Im} P$ and $y_2 \in \operatorname{Ker} P$. Then
$$\begin{aligned} Ax &= A(x_1 + x_2) \\ &= Px_1 + Px_2 + Q(I - P)x_1 + (I - P)Qx_2 \\ &= Px_1 + Py_1 + Py_2 + 0 + (I - P)x_2 \\ &= x_1 + y_1 + 0 + (I - P)(y_1 + y_2) \\ &= x_1 + y_1 + y_2 = x. \end{aligned}$$
Hence $x \in \operatorname{Im} A$. This shows $\operatorname{Im} A = \operatorname{Im} P + \operatorname{Im} Q = E + E'$. □

2. Show that U is a self-adjoint unitary operator if and only if $U = 2P - I$ for some orthogonal projection operator P.

Remark. The question has been modified by adding the assumption that P is orthogonal. Otherwise, an easy counter-example arises in the Hilbert space $\mathbb{C}^2$: Let $P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Then $P^2 = P$. However, $P^* \neq P$ and $2P - I$ is not self-adjoint.

Recall that for a projection P, $\{$ P is orthogonal $\}$ $\iff$ $\{$ P is self-adjoint $\}$ $\iff$ $\{$ P is normal $\}$.

Proof. ($\Longrightarrow$) Set $P := (U+I)/2$. It follows from U being a self-adjoint unitary operator that $U^2 = U^*U = I$. Then

$$P^2 = \left(\frac{U+I}{2}\right)^2 = \frac{U^2+2U+I}{4} = \frac{I+2U+I}{4} = \frac{U+I}{2} = P.$$

Hence P is a projection. Moreover, P is self-adjoint since U is self-adjoint. Then P is an orthogonal projection.

($\Longleftarrow$) Set $U := 2P - I$. Then U is self-adjoint since P is self-adjoint. Moreover, by $P^2 = P$,

$$U^2 = (2P-I)^2 = 4P^2 - 4P + I = 4P - 4P + I = I.$$

Hence $U^*U = UU^* = U^2 = I$. This shows that U is a self-adjoint unitary operator. □

3. Show that U is a self-adjoint unitary operator on a Hilbert space H if and only if there exist orthogonal closed subspaces E_1, E_2 such that H is the direct sum $E_1 \oplus E_2$ and for every $x = x_1 + x_2$ with $x_1 \in E_1, x_2 \in E_2$,

$$Ux = x_1 - x_2,$$

that is, U is a reflection.

Proof. ($\Longrightarrow$) Let $P = (U+I)/2$. Then P is an orthogonal projection by the previous exercise. Then

$$H = \operatorname{Im} P \oplus \operatorname{Ker} P \quad \text{and} \quad \operatorname{Im} P \perp \operatorname{Ker} P.$$

Let $x \in H$. Then $x = x_1 + x_2$ for some $x_1 \in \operatorname{Im} P$ and $x_2 \in \operatorname{Ker} P$. Hence

$$Ux = (2P - I)(x_1 + x_2) = 2x_1 - x_1 - x_2 = x_1 - x_2.$$

($\Longleftarrow$) Since $H = E_1 \oplus E_2$ and $E_2 = E_1^\perp$, there is an orthogonal projection P onto E_1 and an orthogonal projection $I - P$ onto E_2. Note that for $x \in H$,

$$x = Px + (I-P)x,$$

where $Px \in E_1$ and $(I-P)x \in E_2$. Then

$$Ux = Px - (I-P)x = (2P-I)x.$$

Hence $U = 2P - I$, and so U is a self-adjoint unitary operator by the previous exercise. □

— *THE END* —