THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2022-23 Term 1
Solution to Homework 8

1. (a) Let E_1 and E_2 be subspaces of an inner product space. Prove that $E_1 \perp E_2$ if and only if

$$\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2 \tag{1}$$

whenever $x_1 \in E_1, x_2 \in E_2$.

(b) In contrast to part (a), give an example of a Hilbert space H and vectors $x_1, x_2 \in H$ such that $\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$, but $\langle x_1, x_2 \rangle \neq 0$.

Proof. (a) ($\implies$) If $E_1 \perp E_2$, then $\langle x_1, x_2 \rangle = 0$ for $x_1 \in E_1$ and $x_2 \in E_2$. Hence

$$\begin{aligned}
\|x_1 + x_2\|^2 &= \langle x_1 + x_2, x_1 + x_2 \rangle \\
&= \langle x_1, x_1 \rangle + 2\Re\langle x_1, x_2 \rangle + \langle x_2, x_2 \rangle \\
&= \|x_1\|^2 + \|x_2\|^2 + 2\Re\langle x_1, x_2 \rangle \\
&= \|x_1\|^2 + \|x_2\|^2.
\end{aligned}$$

($\impliedby$) Suppose (1) holds for $x_1 \in E_1$ and $x_2 \in E_2$. Then

$$\|x_1 + ix_2\|^2 = \|x_1\|^2 + \|ix_2\|^2 = \|x_1\|^2 + \|-ix_2\|^2 = \|x_1 - ix_2\|^2$$

and similarly,

$$\|x_1 + x_2\|^2 = \|x_1 - x_2\|^2.$$

Hence by Polartization identity,

$$\langle x_1, x_2 \rangle = \frac{1}{4}\left(\|x_1 + x_2\|^2 - \|x_1 - x_2\|^2 + i\|x_1 + ix_2\|^2 - i\|x_1 - ix_2\|^2\right) = 0.$$

Thus $E_1 \perp E_2$.

(b) Consider $\mathbb{C}$ as a complex Hilbert space with the inner product $\langle x, y \rangle := x\overline{y}$. Then

$$\|1 + i\|^2 = 2 = \|1\|^2 + \|i\|^2 \quad \text{but} \quad \langle 1, i \rangle = -i \neq 0.$$

□

2. Let S be a bounded sesquilinear form on $X \times Y$. Define

$$\|S\| := \sup\left\{|S(x,y)| : \|x\| = 1,\ \|y\| = 1\right\}.$$

Show that

$$\|S\| = \sup\left\{\frac{|S(x,y)|}{\|x\|\|y\|} : x \in X \setminus \{0\},\ y \in Y \setminus \{0\}\right\}$$

and

$$|S(x,y)| \leq \|S\|\|x\|\|y\|, \tag{2}$$

for all $x \in X$ and $y \in Y$.

Proof. Denote

$$\|S\|_* := \sup \left\{ \frac{|S(x,y)|}{\|x\|\,\|y\|} : x \in X \setminus \{0\},\ y \in Y \setminus \{0\} \right\}.$$

For $x \in X, y \in Y$ with $\|x\| = 1$ and $\|y\| = 1$, we have $\|S(x,y)\| = \dfrac{|S(x,y)|}{\|x\|\,\|y\|} \leq \|S\|_*$. Hence $\|S\| \leq \|S\|_*$. By the sesquilinearity of S, for $x \in X \setminus \{0\},\ y \in Y \setminus \{0\}$,

$$\frac{|S(x,y)|}{\|x\|\,\|y\|} = |S(\frac{x}{\|x\|}, \frac{y}{\|y\|})| \leq \|S\|$$

since $x/\|x\|$ and $y/\|y\|$ are unit vectors, thus $\|S\|_* \leq \|S\|$. Hence $\|S\| = \|S\|_*$.

This shows that (2) holds for all $x \in X \setminus \{0\},\ y \in Y \setminus \{0\}$. Since S is sesquilinear,

$$\begin{aligned} S(0,y) &= S(0+0,y) = 2S(0,y) \implies S(0,y) = 0 \\ S(x,0) &= S(x,0+0) = 2S(x,0) \implies S(x,0) = 0. \end{aligned}$$

Thus (2) also holds when $x = 0$ or $y = 0$. □

3. Let $T \colon \ell^2 \to \ell^2$ be defined by

$$T \colon (x(1), \ldots, x(n), \ldots) \mapsto (x(1), \ldots, \frac{1}{n}x(n), \ldots)$$

for $x = (x(i)) \in \ell^2$. Show that the range $\mathcal{R}(T)$ is not closed in ℓ^2.

Proof. Suppose on the contrary that $\mathcal{R}(T)$ is closed in ℓ^2. Note that T is injective. It follows from Open Mapping Theorem that the inverse map

$$S \colon \mathcal{R}(T) \to \ell^2,\ (y(1), \ldots, y(n), \ldots) \mapsto (y(1), \ldots, ny(n), \ldots)$$

for $y = (y(i)) \in \mathcal{R}(T)$, is bounded. However, for $n \in \mathbb{N}$, let $e_n = (e_n(i))_{i=1}^{\infty}$ with $e_n(i) = \begin{cases} 1 & i = n \\ 0 & i \neq n \end{cases}$. Then $e_n \in \mathcal{R}(T)$ and $\|e_n\| = 1$. Hence $\|S\| \geq \|Se_n\| = n \to \infty$ as $n \to \infty$, which contradicts the boundedness of S. □

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